# CHAPTER 3: GRAPHS AND OPERATORS OF MARRIAGE THEORY

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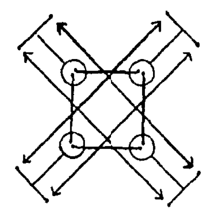
One of the better developed areas of social theoretic mathematics is the use of graphs and relations to describe actual relations among individuals in particular populations. In this chapter, we give specific formulations of the use of graphs and operators to describe particular networks, and to relate these problems to other related problems, such as the prediction of possible future networks from given present states, or to the particular interpretation of various possible measures on the operators themselves or on functions of the operators. We present a summary of the origins in the existing literature of our formalism, then the formalism itself.

Two of the most basic ideas may be traced to Sewall Wright. One of these is the already well known technique of path analysis, which postulates a network of people connected by lines representing descent links, and then gives methods of calculating degree of relatedness along these various descent paths (Wright, 1968). Wright first completed this work early in the century, along with a second important calculation. In a paper in 1921, he showed that the maintenance of a particular system of inbreeding could be associated with a smallest number of people required in each generation to maintain the system. To do this, Wright assumed a rather ideal case of two offspring to each mating, and that these were always one male and one female. These restrictions allowed him to calculate, for example, that where marriage (i.e., mating) to first cousins is prohibited, at least eight people are required in each generation (Wright, 1921).

Using a slight variant of the ordinary diagramming technique, this may be shown as:



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* This paper is an expanded version of the paper "Foundations of Marriage Theory" given by the authors at the 1971 meetings of the American Association of Physical Anthropologists, Boston, Massachusetts. The underlying logic and its potential relation to the Dirac notation was originally outlined in the doctora dissertation of P. A. Ballonoff, but developed into the present form only in co operation with T. Duchamp while both were at Southern Illinois University. See also, Duchamp and Ballonoff (1974).

n this picture, we use a dot "•" to represent a person, whether male or female, a ine between two dots "—" to show that the two are siblings, a circle around two ots to show a "marriage" between them, and a sibling line with a vertical insert nto a marriage circle to show descent from a marriage. The double pointed rrows show available partners across the diagonal of the square are available to ach other.

We call the number of "distinct families" (groups of persons connected by the ame sibling link) required in each generation to maintain the stability of the ule, the structural number (also minimal stability number) for that system. (See hapter 4 for details of this topic.) It is very useful to find such a diagram for ach marriage system one wishes to study. Indeed, this is very close to or identical to the procedure already followed by many ethnographers and ethnologists. For example, R. T. Zuidema has constructed precisely such a system, with seven sides and from four to five layers of generation depth, in order to discuss the lineage system of Incan Peru, and to analyze similarities to North American clan systems (Zuidema, 1964, 1965); E. A. Cook has presented similar arguments for the Manga of New Guinea (1967, 1969); and B. Ruheman has shown similar properties of the classic Australian systems (1945). Another property of these same diagrams is that one may study them more easily as patterns of cycles and chains of an abstract mathematical system. F. B. Livingstone (1969) and C. Levi-Strauss (1967, Chapter 15; 1969) have both indicated this should be a profitable attack.

Next, it has long since been pointed out that related properties of marriage systems may be represented as group operators of various sorts. Andre Weil presented the needed demonstration in the appendix to the 1949 edition of Levi-Strauss (1969) referenced above, subsequently being reprinted in H. White (1963). This later book was also important for its axiomatization of kinship systems, and its mapping of such systems onto role trees suitable for later analysis. In our representation, we also use operators to represent the properties of marriage systems. The notations and methods of analysis we suggest are very similar to those adopted by Maruyama and Yasuda (1970), and by Maruyama (1970), but we have modified this approach by use of a notation similar to that found in Hall and Callins (1971). Use of this notation will provide a very free mathematical framework in which to work, since in the present case, it implies acceptance of the use of Euclidian spaces, easily generalized to Hilbert spaces, as the proper mathematical space for treatment of problems of marriage theory. In such spaces, one may define and calculate long term averages, as well as carry out certain algebraic operations.

### 3.1 Summary of Notation

To make these ideas available to anthropology, we have adopted the "Dirac notation" of "bra" $\langle|$ and "ket" $|\rangle$ to show elements of a Euclidian space. If the basis on the space is chosen to correspond to individuals in a population, the elements may be written $\langle p_i|$ or $|p_i\rangle$. Taking the cross product of the population with itself, and writing marriage kinship rules as logical operations, one may create a square matrix of zeros and ones according to the truth value assignments peculiar to the system under study. We may thus construct self-adjoint linear operators, which might be designated:

$M_t$ : marriage operator in generation t
$S_t$ : same sex designator
$X_t$ : exclusion by "incest prohibition"
${}_jC_i$: generalized cousin operator showing j links of "depth" and i links of width
$B_t$ : sibling operator
etc.

These operators allow demonstration of structural theorems with demographic content. For example, the theorem $tr(M \cdot C) = 0$ implies that a marriage rule "no one is married to a (specific) cousin" holds. The existence of such rules corresponds to the existence of stability rules like: the least number of persons per generation required to maintain first cousin exclusion in marriage is eight. These clearly correspond as well to numbers used for Wright's breeding calculations. (Chapter 4 discusses these numbers in more depth.) Using these operators, and possibly others depending on the society and the marriage rule, we can describe the "marriage history" of a society as a chain of Euclidian spaces and operators, each generation link-d to the preceding one by a progeny transformation:

$$H_o \xrightarrow{P_o} \ldots \; H_t \xrightarrow{P_t} \quad H_{t+1} \longrightarrow \; \ldots$$

where $H_o$ is the Euclidian space of the zeroth generation, $P_o$ the progeny operator from $H_o$ to $H_1$, etc. We define, $N$, to be a link in the chain

$$N_t = \left[\begin{array}{c} H_t \xrightarrow{P_t} H_{t+1} \\ H_t \xrightarrow{B_t} H_t \\ H_t \xrightarrow{M_t} H_t \\ H_t \xrightarrow{({}_iC_j)_t} H_t \end{array}\right]$$

where it is understood that the operators $B_t$, etc. are on $H_t \times H_t$. Then the marriage rule, birth rates, mortality rates, etc., will determine a set of transition probabilities, $\langle N|N_o\rangle$ being the probability that a society described by network $N_o$ will be described by network $N$ in one generation.

We may then define $\langle N|N_o\rangle_t$ as the probability that a society described by $N_o$ will be described by $N$ in t generations. Letting $U$ be the set of all networks, the expression $\langle|N_o\rangle_t$ defines a probability measure on $U$. Suppose $f: U \rightarrow R$ is a function from $U$ to the real numbers. Then f can be considered as a random variable on the probability space $(U, \langle|N_o\rangle_t)$ and

$$\hat{f}_{N_o}(t) = \int f(N) \langle N|N_o\rangle_t$$

will be the expected value of f in n generations after $N_o$. In particular, we define for $N \in U$

$r(N)$ = population of $N$
$\bar{E}(N)$ = average ties of $N$
$\bar{E}_g(N)$ = average genetic ties of $N$
$\bar{E}_m(N)$ = average marriage-sibling ties of $N$
etc.

The expectation values of these functions can be useful. For example, $\hat{r}_{N_o}(t)$ is the expected population in t generations after $N_o$. $\hat{\bar{E}}_{g_{N_o}}(t)$ is the average genetic ties n generations after $N_o$. We note that this is related to the expected average inbreeding coefficient. We may also write more general expressions for inbreeding in a generation and the existence of cycles, as in equations (1).

## 3.2 Detailed Foundations

Consider a society that uses a given marriage rule (see chapters a and 4 for two different definitions of "marriage rule") and which is such that generations may be assumed discrete. We will show how to represent the marriage theory (hereafter termed "history") of the society in terms of linear transformations of a real Euclidean space. Note that in chapter 4 is provided a more compact summary of an application of the following.

### FUNDAMENTAL DEFINITIONS

Definition 1: Let $R$ be the real numbers and let

$$H = \bigoplus_{i=1}^{k} R_i \qquad R_i = R,\ i = 1,2,\ldots,$$

where k is at least as large as the population size, and ususally much larger, but finite and $H$ is a finite dimensional vector space over $R$. We define an inner product on $H$ by

$$\langle A|B\rangle = \sum_{i=1}^{k} a_i b_i$$

where

$$|A\rangle = \bigoplus_{i=1}^{k} a_i \qquad a_i \in R_i$$

$$|B\rangle = \bigoplus_{i=1}^{k} b_i \qquad b_i \in R_i$$

These two bottom sums are formal sums and not sums of real numbers, whereas the first sum is a numerical sum. This inner product is clearly a complete, positive definite, bilinear form on $H$. We stipulate that all Euclidean spaces used will be isomorphic to $H$, unless otherwise specified.

Definition 2: A set $N \subseteq H$ will be called an orthonormal base (or simply a basis) for $H$ if

i) For all $|a\rangle, |b\rangle \in N$, $\langle a|b\rangle = \begin{cases} 1 \text{ if } |a\rangle = |b\rangle \\ 0 \text{ if } |a\rangle \neq |b\rangle \end{cases}$

ii) Any vector in $H$ can be written in a unique way as a linear combination of a finite number of elements of $N$.

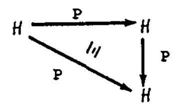
(This definition is more restrictive than the usual definition of orthnormal basis as used in analysis.)

Definition 3: A function $F: H \rightarrow K$, where $H$ and $K$ are Euclidean spaces, is called a linear transformation if for all $|v\rangle, |u\rangle \in H$ and $a, b \in R$,

$$F(a|v\rangle + b|u\rangle) = aF|v\rangle + bF|u\rangle.$$

If $H = K$, then we call F a linear operator (or simply an operator) on $H$.

Definition 4: An operator $P:H \to H$ is called a projection operator if $P$ is the identity on a subspace of and zero on the orthogonal complement of this subspace. Equivalently, $P^2 = P$, which is symbolized by the commutative diagram:

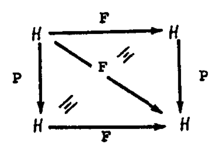



Definition 5: Let $N$ be an orthonormal base of $H$ and $F:H \to H$ an operator on $H$. The trace of F (symbolized by tr(f)) is

$$tr(F) = \sum_{|a\rangle \in N} \langle a|F|a\rangle$$

The trace of an operator is independent of the orthonormal basis chosen.

Definition 6: Let $H$ be a Euclidean space, $P$ a projection operator and $F:H \to H$ an operator on $H$. F is called a P-operator if we have:




that is, the range of F is in the range of P and F is zero on the orthogonal complement.

We may now apply the above tc real people and relations between them. Let

$$G_0 = \{p_1^0, p_2^0, \ldots, p_{pop_0}^0\}$$

$$G_1 = \{p_1^1, p_2^1, \ldots, p_{pop_1}^1\}$$

$$\vdots$$

$$G_n = \{p_1^n, p_2^n, \ldots, p_{pop_i}^n\}$$

The elements of $G_i$ correspond to members of the $i^{th}$ generation. $pop_i^t$ = total population of the $i^{th}$ generation.



Let $H_t$ be a Euclidean space and let

$$N_t \{|p_1^t\rangle, |p_2^t\rangle, \ldots, |p_{pop_t}^t\rangle, |p_{pop_t+1}^t\rangle, \ldots\}$$

be a basis for $H_t$. Let $\pi_t:H_t \to H_t$ be the projection operator defined by:

$$\pi_t : |p_j^t\rangle \longmapsto \begin{cases} |p_j^t\rangle \text{ if } 1 \le j \le pop_t \\ |0\rangle \text{ if } pop_t < j \end{cases}$$

Any vector in the range of $\pi_t$, say $\quad |v\rangle = \sum_{j=1}^{pop_t} a_j |p_j^t\rangle$

corresponds to the subset of $G_i$: $\quad v = \{p_j^t \in G_t | a_j \ne 0\}$

the magnitude of $a_j$ serves as a measure of the stress laid on the $j^{th}$ individual.

For example $\quad |v_1\rangle = |p_1^t\rangle + 4|p_2^t\rangle$

stresses $p_2^t$ whereas $\quad |v_2\rangle = 2|p_1^t\rangle + |p_2^t\rangle$

stresses $p_1^t$ although both $|v_1\rangle$ and $|v_2\rangle$ represent the same subset of $G_i$. Note that by using only 0 and 1 as weights, and interpreting "+" as a set theoretic sum, we obtain an expression describing the elements of a particular set.

We now give an interpretation of $\pi_t$-operators on $H_t$. Let $R:H_t \to H_t$ be a $\pi_n$-operator, we define a relation R on $G_t$ by:

$$\forall a,b \in G_i,\ aRb \Leftrightarrow \langle a|R|b\rangle \ne 0.$$

The magnitude of $\langle a|R|b\rangle$ gives the "strength" of the relation R between a and b.

We may work the other way and define a $\pi_i$-operator on $H_t$ given a relation R on $G_t$: Define $R:H_t \to H_t$ by

$$R : |p_j^t\rangle \longmapsto \sum_{i=1}^{pop_t} r_{ji} |p_i^t\rangle$$

$$r_{ji} = \begin{cases} 1 \text{ if } p_j^t R p_i^t \\ 0 \text{ otherwise} \end{cases} \quad | \le j \le pop_t$$

$$R : |p_j^t\rangle \longmapsto |0\rangle,\ j > pop_t.$$

By allowing the $a_{ij}$'s to differ from 1 and 0, we can incorporate the idea of strength of a relation in our operator R.

## 3.3 Operators of Marriage Theory

### (i) Basic Operators:

In this section, we consider some of the relations that are necessary in marriage theory. Consider $G_t$. We note several important relations on $G_i$.

i) $B_t$: $aB_tb \equiv$ a and b are siblings

ii) $M_t$: $aM_tb \equiv$ a is married to b

iii) $C_t$: $aC_tb \equiv$ a is a first cousin of b

iv) $C_t^x$: $aC_t^xb \equiv$ a is a cross-first cousin of b

v) $C_t''$: $aC_t''b \equiv$ a is a parallel-first cousin of b

vi) $S_t$: $aS_tb \equiv$ a is of same sex as b

where a, b $\varepsilon$ $G_i$. Following the last section, we have corresponding to i) - vi), the following operators:

i) $B_t : H_t \to H_t$

$$|p_i^t\rangle \longmapsto \sum_{j=1}^{pop_t} b_{ij}|p_j^t\rangle, \quad b_{ij} = \begin{cases} 1, & \text{if } p_i^t B_t p_j^t \\ 0, & \text{if not } i \leq pop_t \end{cases}$$

$$|p_i^t\rangle \longmapsto |0\rangle, \quad i > pop_t$$

ii) $M_t : H_t \to H_t$

$$|p_i^t\rangle \longmapsto \sum_{j=1}^{pop_t} m_{ij}|p_j^t\rangle, \quad m_{ij} = \begin{cases} 1, & \text{if } p_i^t M_t p_j^t \\ 0, & \text{if not } i \leq pop_t \end{cases}$$

$$|p_i^t\rangle \longmapsto |0\rangle, \quad i > pop_t$$

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vi) $S_t : H_t \to H_t$

$$|p_i^t\rangle \longmapsto \sum_{j=1}^{pop_t} s_{ij}|p_i^t\rangle, \quad s_{ij} = \begin{cases} 1, & \text{if } p_i^t S_t p_j^t \\ 0, & \text{if not } i \leq pop_t \end{cases}$$

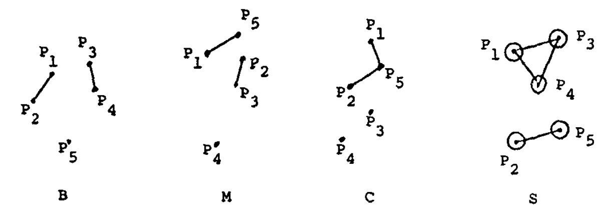
$$|p_i^t\rangle \longmapsto |0\rangle, \quad i > pop_t$$

These are sufficient to describe the "American System" and many others, however, particular marriage systems could require more or even a different set of operators



In this event, one could proceed in a way similar to what follows. Note that given a specific history, these relations are defined empirically. However, we also note that the "kinship logic" of a particular system also restricts which empirical relations may be possible. Our system is capable both of describing empirical networks and making computations on them, and also of creating and studying properties of synthetic or theoretical systems.

We now give an example of how to construct some of the above operators given a particular situation: Suppose $G = \{p_1, p_2, p_3, p_4, p_5\}$ and we had relations corresponding to these graphs:




Then

$$[b_{ij}] = \begin{bmatrix} 0&1&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&1 \\ 0&0&0&0&0 \end{bmatrix} \qquad [m_{ij}] = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&1&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \end{bmatrix}$$

$$[c_{ij}] = \begin{bmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&1&0&0&0 \end{bmatrix} \qquad [s_{ij}] = \begin{bmatrix} 1&0&1&1&0 \\ 0&1&0&0&1 \\ 1&0&1&1&0 \\ 1&0&1&1&0 \\ 0&1&0&0&1 \end{bmatrix}$$

From these matrices, we may compute the values of the operators for any vector. For example, let $|a\rangle = |p_1\rangle + |p_3\rangle$, then

$$B|a\rangle = B|p_1\rangle + B|p_3\rangle = |p_2\rangle + |p_4\rangle$$

$$M|a\rangle = M|p_1\rangle + M|p_3\rangle = |p_5\rangle + |p_2\rangle$$

$$C|a\rangle = C|p_1\rangle + C|p_3\rangle = |p_5\rangle + |0\rangle = |p_5\rangle$$

$$S|a\rangle = S|p_1\rangle + S|p_3\rangle = |p_1\rangle + |p_3\rangle + |p_4\rangle + |p_3\rangle + |p_4\rangle$$

$$= |p_1\rangle + 2|p_3\rangle + 2|p_4\rangle$$

(ii) Some Theorems

The following theorems follow immediately from the nature of relations i) - vi).

Theorem 1: For t = 0, 1, 2, 3.

$$\begin{array}{ccc} & H_t/\mathrm{Ker}(B_t+\pi_t) & \\ \zeta_n \nearrow & & \searrow \\ H_t & \xrightarrow{\;C_t\;} & H_t \end{array} \qquad \begin{array}{ccc} & H_t/\mathrm{Ker}(B_t+\pi_t) & \\ \zeta_n \nearrow & & \searrow \\ H_t & \xrightarrow{\;C''_t\;} & H_t \end{array} \qquad \begin{array}{ccc} & H_t/\mathrm{Ker}(B_t+\pi_t) & \\ \zeta_n \nearrow & & \searrow \\ H_t & \xrightarrow{\;C^x_t\;} & H_t \end{array}$$

where $\mathrm{Ker}(B_t + \pi_t) = \{|v\rangle \in H_t : (B_t + \pi_t)|v\rangle = 0\}$ and $\zeta_t$ is the natural map onto the quotient space. Notice that dimension $(H_i/\mathrm{Ker}(B_i + \pi_i)$ = number of distince families in G.

Theorem 2: For t = 0,1,2,..., $C_t = C^x_t + C''_t$.

Theorem 3: For t = 0,1,2,..., $B_t M_t C_t, C^x_t, C''_t, S_t$ are self adjoint operators.

Theorem 4: For t = 0,1,2,..., $F \in \{B_t, M_t, C_t, C^x_t, C''_t\}$, tr(F) = 0.

Theorem 5: For t = 0,1,2,..., and Axioms corresponding to marriage rules A1 to A5 of the next chapter.

Axiom 1) $\longleftrightarrow$ $\mathrm{tr}(M_t) = 0$.

Axiom 2) $\longleftrightarrow$ $\mathrm{tr}(M_t \cdot B_t) = \mathrm{tr}(M_t) = 0$.

Axiom 3) $\longleftrightarrow$ $\mathrm{tr}(M_t \cdot C''_t) = \mathrm{tr}(M_t \cdot B_t) = \mathrm{tr}(M_t) = 0$.

Axiom 4) $\longleftrightarrow$ $\mathrm{tr}(M_t \cdot C^x_t) = \mathrm{tr}(M_t \cdot B_t) = \mathrm{tr}(M_t) = 0$.

Axiom 5 $\longleftrightarrow$ $\mathrm{tr}(M_t \cdot C_t) = \mathrm{tr}(M_t \cdot B_t) = \mathrm{tr}(M_t) = 0$.

(Note: $\mathrm{tr}(M_t \cdot C_t) = 0 \longrightarrow \mathrm{tr}(M_t \cdot C''_t) = \mathrm{tr}(M_t \cdot C^x_t) = 0$.)

These theorems on zero trace of particular operators under particular marriage rules suggests the existence of a theory which studies non-zero trace operators or products and powers of operators. In fact, non-zero traces correspond to the existence of cycles and chains of various length of individuals connected through the operators (or products or powers of operators) whose traces are taken. In the present example, the results simply hold from the fact that a particular marriage rule is followed in a given case.

(iii) Progeny Transformation

We now present a transformation relating people in one generation to their children in the next.



Definition 7: For t = 0,1,2,..., the $i^{th}$ progeny transformation is a map $\mathrm{Prog}_t : H_t \to H_{t+1}$

$$|p^t_j\rangle \longmapsto \sum_{k=1}^{pop_{t+1}} p_{jk}|p^{t+1}_k\rangle, \quad 1 \le j \le pop_t$$

$$p_{jk} = \begin{cases} 1, & \text{if } p^t_j \text{ is a parent of } p^{t+1}_k \\ 0, & \text{otherwise} \end{cases}$$

$$|p^t_j\rangle \longmapsto |0 \quad \text{if } j > pop_t$$

Since $\mathrm{Prog}_t$ is zero outside the image of $\pi_t$ and has image in the image of $\pi_{t+1}$, we have:

Theorem 7: For t = 0,1,2,...,

$$\begin{array}{ccc} H_t & \xrightarrow{\;\mathrm{Prog}_t\;} & H_{t+1} \\ \pi_t \downarrow & \searrow & \downarrow \pi_{t+1} \\ H_t & \xrightarrow[\;\mathrm{Prog}_t\;]{} & H_{t+1} \end{array}$$

Theorem 8: If marriage is monogamous, then for t = 0,1,2,..., we have:

$$\begin{array}{ccc} & & H_t/\mathrm{Ker}(M_t + \pi_t) \\ & \zeta_t \nearrow & \downarrow \\ H_t & \xrightarrow{\;\mathrm{Prog}_t\;} & H_{t+1} \end{array}$$

where $\mathrm{Ker}(M_t + \pi_t) = \{|a\rangle \in H_t : (M_t + \pi_t)|a\rangle = 0\}$ and $\zeta_t$ is the natural map.

Theorem 9: For t = 0,1,2,...,

$$\langle p^{t+1}_j | B_{t+1} | p^{t+1}_k \rangle \neq 0 \longrightarrow$$

$$\mathrm{Prog}^t_h | p^{h+1}_j \rangle = \mathrm{Prog}^t_h | p^{h+1}_k \rangle$$

Notice that the operator $\mathrm{Prog}_t$ here can be constructed from the descent maps of the next chapter. Similarly, $pop_t$ is simply the size of the set $G_t$, which the operators M, B, S here correspond to the partitions M, B, S of the next chapter, and G, the set of people of a given generation remains the same concept. We use "P" here for an arbitrary descent transformation, and $\mathrm{Prog}_t$ for the specific transformation from t to t+1. Note that P is also a projection. A similar looseness occurs by using $N$ in parts to refer to a specific network, and sometimes to refer to all networks isomorphic to the specific network. The context is generally clear for networks, but see section 4.3 for a clearer treatment of the difference between networks and isomorphism classes of networks.

(iv) Marriage History

We now have the apparatus to describe the history of a society. For each generation, $G_t$, we have a Euclidean space $H_t$, an orthonormal basis $N_t$, a projection operator $\pi_t$, several operators: $M_t$, $B_t$, $C_t$, $S_t$, etc., (the exact operators in the list depending on the marriage system under consideration) and a progeny transformation, $Prog_t$. We also have the conditions the operators must satisfy to be compatible with reality (some of these conditions are given in the previous theorems). Thus, we have schematically:

$$\begin{array}{ccccccc}
G_0 & G_1 & G_2 \cdots & G_t & & G_{t+1} & \\
H_0 \longrightarrow & H_1 \longrightarrow & H_2 \cdots \to & H_t \xrightarrow{Prog_t} & & H_{t+1} \longrightarrow & \cdots \\
H_0 \xrightarrow{\pi_i} H_0 & & \cdots & H_t \xrightarrow{\pi_t} H_t & & & \\
H_0 \xrightarrow{M_i} H_0 & & \cdots & H_t \xrightarrow{M_t} H_t & & & \\
H_0 \xrightarrow{B_i} H_0 & & \cdots & H_t \xrightarrow{B_t} H_t & & & \\
\vdots & & & \vdots & & &
\end{array}$$

A diagram as above corresponds to a specific history of a society. This cannot be predicted. However, probably diagrams can be found and their probabilities calculated. This is what we seek to do in what follows.

## 3.4 Relation Strength

In sections 3.2 and 3.3, we found out how to represent relationships as linear transformations. In this section, we develop some operators which measure the strength of the marriage rule. This measure is useful in an analysis of the genetics and economics of a society.

In this section, B, M and P denote the sibling, marriage and progeny operators on a generation. Also we suppress subscripts as generation numbers needed in formulas



will be obvious from context. Also, we may assume an infinite history for a society by augmenting the diagram so that for $t < 0$, the history is empty. We always are referring to the case in what follows unless otherwise specified. We assume that we are working in particular generation with finite population size g.

(i) Generalized Cousin Operator

Definition 10: Let $(H,N,\pi)$, $(H',N',\pi')$, and $(H'',N'',\pi'')$ be three objects in the set of all triples (Euclidean spaces, possible networks, populations of individuals), with $N = \{|p_1\rangle, |p_2\rangle, \ldots$ , The same for $N'$ and $N''$ except with primes; with

$$\pi:\ |p_i\rangle \longmapsto |p_i\rangle \quad 1 \le i \le g$$

$$|p_i\rangle \longmapsto |0\rangle \quad g < i$$

and the same for $\pi'$ and $\pi''$ with g' and g" in place of g. Let $F{:}H \to H'$, $G{:}H' \to H''$ be two linear transformations defined by the matrices: $F = [f_{ij}]_{g'xg'}$ $G = [g_{ij}]_{g''xg'}$ where $f_{ij} \ge 0$, $g_{ij} \ge 0$ for all i, j. Then define G*F by:

$$G{*}F{:}\ H \longrightarrow H''$$

$$|p_i\rangle \longmapsto \sum_{j=1}^{t''} a_{ij}|p_j''\rangle$$

where $$a_{ij} = \begin{cases} 1 \text{ if } & p_j''|G{\cdot}F|p_j\rangle \ne 0 \\ 0 \text{ otherwise} & \end{cases}$$

where $G{\cdot}F$ is composition of functions. In the following, all multiplications of operators will be *-multiplication.

Theorem: If all operators are as in Definition 10, then * is associative.

Note: Denote $\underbrace{G{*}G{*}\ldots{*}G}_{k\text{-times}}$ by $G^{k*}$, and denote the transformation of an operator P by $P^T$.

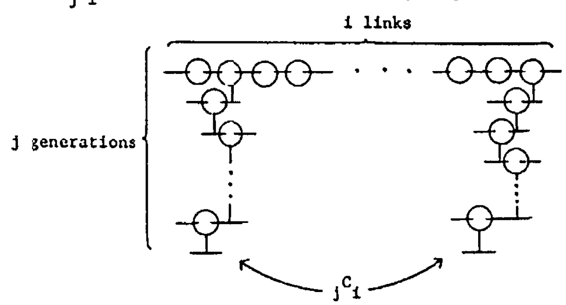
Definition 11: Consider the dieagram below. Assume we are in a particular generation j, say the $t^{th}$ generation with population size g. We define operators on $H_t$:

$${}_jC_0 = (P^{j*})*(B)*(P^T)^{j*} \qquad 1 \le j \le \infty$$

$${}_jC_i = (P^{j*})*(B + M)^{i*}*(P^T)^{j*} \qquad 0 \le j \le \infty,\ 1 \le i$$

Note that if $j > q$, then ${}_jC_i = 0$.

We call these operators "generalized cousin operators". When it becomes necessary to show generation numbers, we write ${}^{q}_{j}C_{i}$ (q = generation number). The relations described by the ${}_{j}C_{i}$ are illustrated by the following diagram:




Thus, ${}_{1}C_{1}$ is the "first cousin operator", ${}_{2}C_{1}$ is the "second cousin operator".

Note: Since $P^T * P \neq 1$, we do not have ${}_{K}C_{S} * {}_{K}C_{T} = {}_{K}C_{S+T}$.

The set $E = \{{}_{i}C_{j}\}$ describes paths that genes could follow as well as some measure of economic and political ties. We now seek a measure of the strength of relational ties between individuals ("bond strength").

Definition 12: Let $E = aI + bB + cM + c_m E_m + c_p E_p + c'E'$

where $c_m$, $c_p$, $c'$ are real numbers $\geq 0$, and

$$E_m = \sum_{k=1}^{\infty} (1/2k)\ {}_{0}C_{K}, \quad E_g = \sum_{k=1}^{\infty} \left(\frac{1}{2^{2j+1}}\right) {}_{j}C_{1}, \quad E' = \sum_{\substack{j=1 \\ k=2}}^{\infty} \left(\frac{1}{2^{2j+k-1}}\right) {}_{i}C_{j}$$

These operators show that $E_g$, the usual counting function for measuring probability of identity by descent, can be simply stated as an analytic form using the previous definitions; and that other particular networks of relations among individuals may be stated and computed with various weights. In the present case, we pick 1/2 to specified powers as the weight to emphasize correspondence to genetic theoretic probabilities, but other weights for $E_m$, may also be selected.

Given two subsets of a generation $\{|p_{i_1}\rangle, |p_{i_2}\rangle, \ldots |p_{i_r}\rangle\}$ and $\{|p_{j_1}\rangle, |p_{j_2}\rangle \ldots |p_{j_s}\rangle\}$ we can define the average bond strength between individuals in these two sets. Let $|p_{i_1} \cdots p_{i_r}\rangle = \sum_{k=1}^{r} |p_{r_k}\rangle$ and $|p_{j_1} \cdots p_{j_s}\rangle = \sum_{k=1}^{s} |p_{j_k}\rangle$. Then

$$\frac{\langle p_{i_1} \cdots p_{i_r} | E | p_{j_1} \cdots p_{j_s} \rangle}{\langle p_{i_1} \cdots p_{i_r} | p_{i_1} \cdots p_{i_r} \rangle \langle p_{j_1} \cdots p_{j_s} | p_{j_1} \cdots p_{j_s} \rangle}$$

is a measure. By $|E|$ we mean:

$$\frac{\langle p_1 \cdots p_g | E | p_1 \cdots p_g \rangle}{(\langle p_1 \cdots p_g | p_1 \cdots p_g \rangle)^2}$$

where g is the total population size.

### 3.5 Probabilistic Formulation

In section 3.2 we showed how the marriage history of a society can be represented by a diagrammatic sequence of operators, which in fact is a statement of the history of a society. Starting with the initial state, an infinite number of diagrams are possible, each corresponding to a particular history. However, some diagrams are more probable than others. What we wish to do is discuss the formulation of probabilities of different possible histories.

Consider a population $G = \{p_1, \ldots, p_g\}$ of g people, represented as a space $H$ with basis $N = \{|p_1\rangle, \ldots |p_g\rangle, \ldots\}$ and projection operators $\pi: H \to H$. Let $B: H \to H$ be the sibling operator. In a society which has a marriage rule which specifies whom a person cannot marry (or equivalently, whom a person can marry) as a set of people which are in one or more particular relations to that person we may characterize the marriage rule as follows: there is a set of $\pi$-operators $E = \{R^1, \ldots R^k\}$ on $H$. The marriage operator $M: H \to H$ must satisfy the condition "$\langle p_i | M | p_j \rangle = 1$" (that is, $p_1$ may marry $p_j$) only if $\langle p_i | F | p_j \rangle = 0$ for all $F \in E$. (Or, $p_i$ is not related to $p_j$ by any of the relations represented by $E$.) This may be stated in terms of an "exclusion operator".

Definition 13: Let $E$ be as above, $X: H \to H$ is an operator defined for $|p_i\rangle \in N$ by:

$$1 \leq i \leq g \quad X|p_i\rangle = \sum_{j=1}^{n} e_{ij} |p_j\rangle$$

$$e_{ij} = \begin{cases} 1 \text{ if there is an } F \in E \text{ such that } \langle p_j | F | p_i \rangle \neq 0. \\ 0 \text{ otherwise.} \end{cases}$$

$$g < i \quad X|p_i\rangle = |0\rangle$$

The marriage rule may be stated as follows: $tr(X \cdot M) = 0$
for any admissible marriage operator.

Note that the operators which compose to form the exclusion operator are precisely those representing classes of relations which the marriage states as prohibited. Note also that any rule, whether stated initially in a proscriptive or prescriptive form, may be translated into a statement of which types of relations are prohibited as partners, and which are not. The composition used here is to combine by a logical "or" all those operators representing excluded classes under a given rule.

We assume that all marriages in a generation take place sequentially in a relatively short span of time, that monogamy is practiced, and that marriages continue until all persons marry that can without violating the marriage rule. With the assumption the reader will note that the set of allowable marriage sequences is in 1-1 correspondence with the following set, labeled $X\mathcal{M}(g)$:

Definition 14: Let X be the exclusion operator on a population of size g as described above. Let $s$ = greatest integer $\leq g/2$. Then $X\mathcal{M}(n)$ is the set of s-triples of g x g matrices of "0's" and "1's" such that $(M_1, M_2, \ldots, M_s) \in X\mathcal{M}(g)$ if and only if the following hold:

i) $M_i$ is symmetric and has zero's along the diagonal for $1 \leq i \leq s$.

ii) There is at most one "1" in every row of $M_i$ for $1 \leq i \leq s$. (This is the monogamy assumption.)

iii) If we consider X as an g x g matrix (or $M_i$ as an operator) the diagonal of the matrix $X \cdot M_i$ (or the trace of the operator $X \cdot M_i$) is zero for $1 \leq i \leq s$. (This is the marriage rule.)

iv) The number of "1's" in $M_i$ is equal to the number of "1's" in $M_{i+1}$ or the number of "1's" in $M_{i+1} - 2$.

v) If (the number of "1's" in $M_i$) = (the number of "1's" in $M_{i+1}$), then $M_i = M_{i+1}$ for $1 \leq i \leq s - 1$.

vi) If $M_i$ is the first matrix in $(M_1, \ldots, M_s)$ such that $M_i = M_{i+1}$. Then if M' is any g x g matrix with a "1" in the j, k position, if $M_i$ has a "1" in the j, k position, and if M' has a "1" in a position where $M_i$ does not have a "1", then M' violates i), ii), or iii). (This states that all people that can marry do so.) We call $X\mathcal{M}(g)$ the "marriage space of X".

We now define a probability measure on $X\mathcal{M}(g)$:



Definition 15: PX is the probability measure on $X\mathcal{M}(g)$ defined by:

$$\forall A \in X\mathcal{M}(g) \qquad PX(A) = \frac{1}{|X\mathcal{M}(g)|} \quad ,$$

$$\forall S \subseteq X\mathcal{M}(g) \qquad PX(S) = \sum_{A \in S} XP(A).$$

where $|X\mathcal{M}(g)|$ is the number of elements in $X\mathcal{M}(g)$. This is the measure gotten if all possible marriage sequences are assumed equally likely. This is not exactly true in general, especially when a segment of the population for some reason has priorities over the rest in choice of marriage partners (for example, rulers of society). Also, there will be sections of the population considered less desirable than others. At this stage in our research, we are content with the above measure which in many cases is close to reality, and which can always be altered should it prove inadequate.

Consider now a population of n people on which are imposed two marriage rules, X and X'. These rules will in general restrict the marriage sequences differently. We may compare X and X' in several ways, some of which will be presented later. For the present, we consider the entropy functions as defined by Shannon and Weaver (1964).

Definition 16: Let $X\mathcal{M}(g)$ be a marriage space and PX the probability measure on it. The entropy of PX, written H(PX), is defined by:

$$H(PX) = -\sum_{A \in X\mathcal{M}(g)} PX(A) \log_2(PX(A)).$$

This function simplifies when PX is defined as above:

$$H(PX) = \log_2 |(X\mathcal{M}(g))|.$$

Essentially H(PX) is a measure of the constraint placed on the population by the marriage rule: the smaller the H(PX), the more constraint. We may now compare X and X' by computing H(PX) and H(PX'). One should note that the exclusion operator is not only a function of the marriage rule, but also of the particular state of the society, hence so is H. For example, a population of 10 unrelated people has a high entropy under first cousin exclusion, but a population of 10 first cousins has a zero entropy.

One author of this chapter (P.A.B.) believes that "entropy" measures to base 2 are not the correct functions. Instead, measures to the base of the population size are more meaningful, and can allow "entropy" to either increase or decrease as a more accurate summary statement of the amount of structure on a population.

However, P.A.B. also believes that entropy measures in either form would be less useful than a theory of "distances" between populations defined in terms of the various possible traces of operators. In such a case, the probabilities used in the "entropy" formulation might also enter as weights, but in a way more useful to the comparison of concrete networks and the evolution of possible networks.

(ii) Progeny

Suppose now that the people of generation G have married in a particular sequence: $(M_1, M_2, \ldots, M_s) = A$ consisting of q marriages. Let $\{m_1, m_2, \ldots m_q\} = C_q$ be the set of couples in order of marriage. We are interested in the progeny of these couples. Suppose $m_i$ has $f_1(m_i)$ males and $f_2(m_i)$ females. Then we have a function:

$$f: \{m_1, \ldots, m_q\} \to Z^+ \times Z^+$$

$$m_i \stackrel{q}{\longmapsto} (f_1(m_i),\ f_2(m_i))$$

where $Z^+ \times Z^+$ is the set of pairs of non-negative integers. We can consider f as a function from $\{1,2,\ldots,q\}$, if we remember the order of marriage. Thus, the set of all functions $f:\{1,2,3,\ldots,q\} \to Z^+ \times Z^+$ corresponds to the set of all possible birth distributions of the couples. The progeny operator: Prog (labeled P) is always to be understood as a direct sum:

$$\text{Prog} = P_m \oplus P_f$$

$P_m$ = male progeny, $P_f$ = female progeny

with components $P_m$ and $P_f$ known. Thus, Prog is really two operators, a male part and a female part.

Definition 17: Let $B_q = \{f:\{1,2,\ldots,q\} \to Z^+ \times Z^+\}$. For $f \in B_q$, let $f_1, f_2$, and $\bar{f}$ be defined by

$$f(i) = (f_1(i), f_2(i)), \quad 1 \le i \le q$$

$$\bar{f}(i) = f_1(i) + f_2(i).$$

Now let $\bar{P}(g)$, $g \in Z^+$, be the probability that a couple will have exactly k children which grow up to mature and reproduce. Let $P_1$ be the probability that a particular child will be a male and $P_2 = 1 - P_1$ the probability that it will be a female.



Then we have a probability measure on $B_q$:

Definition 18: $PB_q$ is the probability measure on $B_q$ defined by:

For $f \in B_q$ $$PB_q(f) = \prod_{i=1}^{q} \bar{P}(\bar{f}(i)) \cdot \binom{\bar{f}(i)}{f_1(i)} P_1^{f_1(i)} P_2^{f_2(i)}$$

Note: If $\bar{P}$ is assumed to be Poisson with mean $\lambda$ we get

$$PB_q(f) = \frac{e^{-q\lambda}\ (\lambda P_1)^{\bar{n}_{f_1}}\ (\lambda P_2)^{\bar{n}_{f_2}}}{\prod_{i=1}^{q} (f_1(i)!)(f_2(i)!)}$$

where $$\bar{n}_{f_1} = \sum_{i=1}^{q} f_1(i),\quad \bar{n}_{f_2} = \sum_{i=1}^{q} f_2(i).$$

By defining $PB_q$ in this way, we can account for the death rate, birth defects, and the like which affect the size of the portion of the actual population which is capable of reproduction. This we call the "breeding population". Thus, if there are 50 people in a population and half are non-reproductive, we say the "breeding population" is 25. (Note that this is "approximately" the "effective population size" of population genetic theory.)

(iii) Case of Several Generations

What has been said about the marriage history of a society can be represented by a list, for each generation of all the functions and operators in that generation together with the properties:

a) $\pi_t$ composed with any of the other operators is that operator.
b) all operators are symmetric.
c) $X_t$ is defined in terms of the $R_t^i$ as above.
d) $E_t$ is defined as in section 3.4 in terms of $\pi_t$, $B_t$, $M_t$ and the ${}_i^tC_j$'s.
e) ${}_0C_j = (B + M)$.

The above analysis leads to an abstract definition of a generation as an ordered tuple:

$$N_t = (H_t, N_t, \pi_t, B_t, M_t, P_t, E_t, X_t, R_t^1, R_t^2, \ldots, R_t, \{{}_i^tC_j\}).$$

Satisfying a), b), c), d), e) when $N_t$ is a basis compatible with $\pi_t$ as in section 3.2. All other conditions imposed by the physical meanings of the operators (e.g., $trM_t = 0$, $tr(M_t \cdot X_t) = 0$, etc.) are also assumed.

Let $\mathcal{U}$ be the set of all such tuples. What we seek is an answer to the question: "Suppose $N_0 \epsilon \mathcal{U}$ is a generation, what is the probability that the progeny of $N_0$ marry and reproduce to produce generation $N \epsilon \mathcal{U}$?"

Definition 19: Let $N_0, N \epsilon \mathcal{U}$, denote by $\langle N|N_0\rangle$ the above probability. Then we have a probability measure on $\mathcal{U}$: $\langle *|N_0\rangle : \mathcal{U} \to R$

$$N \longmapsto \langle N|N_0\rangle$$

We call this the probability of $N_0$ producing $N$ in one generation.

Definition 20: Let $\langle *|N_0\rangle$ be as above, then $\langle *|N_0\rangle_t : \mathcal{U} \to R$ is a probability measure on $\mathcal{U}$ defined by

$$\langle N|N_0\rangle_t = \sum_{\substack{\text{all } (t+1)\text{-tuples} \\ (N_0,\ldots,N_n)\epsilon\mathcal{U}^t \\ N_0=N_0, N_n=N}} \prod_{i=1}^{t} \langle N_i|N_{i-1}\rangle$$

We call this the probability of $N_0$ producing $N$ in t-generations.

Note: $\langle\,|N_0\rangle_1 = \langle\,|N_0\rangle$.

We now proceed to characterize the calculation of $\langle\,|N_0\rangle$:

Let $\quad N_0 = (N_0, H_0, \pi_0, B_0, M_0, P_0, E_0, X_0, R_0', \ldots, R_0, \{{}_i^0C_j\})$

$\quad N = (H, N, \pi, B, M, P, E, X, R', \ldots, R^{\zeta}, \{{}_iC_j\})$

There are several conditions that $N$ must meet for a transition from $N_0$ to $N$ to even be possible. These conditions all follow from the definition of the opeartors and are easily seen to be necessary:

a') $B + \pi = P * P^{-1}$

b') ${}_{i+1}C_j = P * {}_i^0C_j * P_0^{-1} \quad i = 1,2,3,\ldots \quad j = 1,2,3,\ldots$

c') $P\pi_0 = P$

d') Since X, R', ..., $R^{\zeta}$ depend only on $N_0$ (as stated elsewhere in this chapter), they must agree with $N_0$ (the exact statement of how to determine this agreement depends on the particular marriage rule used).

If any of these conditions is not met, $\langle N|N_0\rangle = 0$. If these are met, then once M and P are found, $N$ is completely determined. Let $\pi = \dim I_m(P_0)$ = number of progeny of $N_0$, then we have a measure $PX_0$ on $X_0\mathcal{M}(\pi)$, the space of possible marriage outcomes. Given a marriage outcome:

$$A = (M_1, M_2, \ldots, M_s), \text{ (s = greatest integer} \leq r/2)$$

consisting of $n_A$ marriages, we have a probability measure $PB_{n_A}$ on the space of progeny outcomes, $B_{n_A}$. Given a pair (A,f), $A \epsilon X_0\mathcal{M}(\pi)$, $f \epsilon E_{n_A}$, M and P are determined. Thus, the probability of getting M and P which we label P(M,P) is

$$P(M,P) = \sum_{\substack{\text{sum over} \\ \text{all (A,B) which} \\ \text{produce (M,P).}}} PX_0(A)\, PB_{t_A}(f)$$

But we now have $\quad \langle N|N_0\rangle = q(N_1N_0) \times P(M,P)$

where $\quad q(N_1N_0) = \begin{cases} 1 \text{ if a'), b'), c') and d') are met} \\ 0 \text{ if not.} \end{cases}$

(iv) Random Variables

Now that we have a set of probability measure on $\mathcal{U}$, $\{\langle *|N_0\rangle_t | N_0 \in \mathcal{U}, t = 1,2,3,\ldots\}$ we can begin an analysis of how a marriage system affects the history of a society. This is done in terms of random variables defined on $\mathcal{U}$.

We conclude the chapter by summarizing some useful random variables on $\mathcal{U}$. Let $N = (H, N, \pi, B, M, P, E, X, R', \ldots, R^{\zeta}, \{{}_iC_j\}) \in \mathcal{U}$:

i) $r(N) = \dim(I_m P)$ = number of progeny of $N$.

ii) $\bar{E}(N) = \langle E\rangle$ = average tie of $N$ (see Section 3.3).

iii) $\bar{E}_m(N) = \langle E_m\rangle$ = average marriage sibling ties.

iv) $\bar{E}_p(N) = \langle E_p\rangle$ = average genetic ties.

v) $\bar{E}_s(N) = tr(E)/\pi(N)$ = average self-ties.

Given $N_0 \in \mathcal{U}$ we can use those random variables to find their expected values after t-generations:

Definition 22: Let $N_0 \in \mathcal{U}$, then define:

i) $N_0\langle r\rangle_t = \int_{\mathcal{U}} r(N) \langle N\, N_0\rangle_t$

ii) $N_0\langle \bar{E}\rangle_t \quad \int_{\mathcal{U}} \bar{E}(N) \langle N\, N_0\rangle_t$

iii) $N_0\langle \bar{E}_m\rangle_t = \int_{\mathcal{U}} \bar{E}_m(N) \langle N\, N_0\rangle_t$

iv) $N_0\langle \bar{E}_p\rangle_t = \int_{\mathcal{U}} \bar{E}_p(N) \langle N\, N_0\rangle_t$

v) $N_0\langle \bar{E}_s\rangle_t \quad \int_{\mathcal{U}} \bar{E}_s(N) \langle N\, N_0\rangle_t$

These expectation values define functions as follows:

Definition 23: Let $Z^+ = \{1,2,3,\ldots\}$

i) $r : U \times Z^+ \longrightarrow R$

$(N_0,t) \longmapsto N_0\langle r \rangle_t$

ii) $\bar{E} : U \times Z^+ \longrightarrow R$

$(N_0,t) \longmapsto N_0\langle \bar{E} \rangle_t$

iii) $\bar{E}_m : U \times Z^+ \longrightarrow R$

$(N_0,t) \longmapsto N_0\langle \bar{E}_m \rangle_t$

iv) $\bar{E}_p : U \times Z^+ \longrightarrow R$

$(N_0,t) \longmapsto N_0\langle \bar{E}_p \rangle_t$

v) $\bar{E}_s : U \times Z^+ \longrightarrow R$

$(N_0,t) \longmapsto N_0\langle \bar{E}_s \rangle_t$

These functions serve as predictions of the path a society is likely to take from an initial state.