APPENDIX I

THEORY OF MINIMAL STRUCTURES

I. Introduction

Minimal structures are the fundamental conceptual objects upon which much population genetics and social anthropological theory is based. They describe mating systems as discussed since Wright (1921) in genetics, and "kinship systems" in the tradition of Levi-Strauss (1969). Zuidema (1965) has described them in ideological systems expressed through marriage rules in North American groups, E. A. Cook has described them as "marriage rules" for the New Guinea Manga, B. Ruheman (1945) and F. B. Livingstone (1959) have described them for native Australian marriage rules.

Because of this breadth of application of minimal structures, I have chosen to call the general topic "marriage theory." Subsequent presentations under this rubric will include a linear operator formalism which allows computations on empirical data (Duchamp, 1972). However, the embedding of the theory of minimal structures in the contexts of genetic and social theoretic applications should not obscure the fact that the theory has an independent mathematical existence in its own right.

II. Definitions

DEFINITION 1: A generation $\underline{G}$ is a set of elements p, q, . . . called individuals A configuration $\underline{C}$ is a triple ($\underline{B}$,$\underline{M}$,$\underline{S}$) such that:

(i) $\underline{B}$ is a partition of $\underline{G}$;

(ii) $\underline{M}$ is a collection of two element subsets of $\underline{G}$;

(iii) There are subsets of $\underline{G}$ labeled m and f such that $|m \cap f| = 0$ and $|m \cup f| = |\underline{G}|$ and $\underline{S}$ is the partition of $\underline{G}$ imposed by these labelings:

(iv) $|m \cap M| = 1$ and $|f \cap M| = 1$ for all $M \in \underline{M}$.

In anthropological applications, $\underline{G}$ is interpreted as a population of people of a particular "generation," whose parents (by Def. 2 below) are members of the previous generation and whose descendants are of the following age set. Thus generations are "discrete." The position $\underline{B}$ sorts individuals into sets of siblings who are descendents of the same parents. $\underline{M}$ is a monogamous "marriage" or "mating" relation between individuals of different sexes. Adaptations, sequential marriages, etc. are not allowed by these definitions.

DEFINITION 2: A descent relation d from a generation $\underline{G}$ with a configuration $\underline{C}$ to a generation $\underline{G}'$ with configuration $\underline{C}'$ is a map $d:\underline{B}'\rightarrow M$, which is 1-1 into $\underline{M}$ and such that the inverse map is 1-1, from a subset of $\underline{M}$ onto $\underline{B}'$.

An <u>infinite sequence of generations</u> $\{\underline{G}^{(n)}\}$ is a sequence ... $\underline{G}^{(n)}$, $\underline{G}^{(n+1)}$, ... of generations together with configurations ... $\underline{C}^{(n)}$, $\underline{C}^{(n+1)}$, ... and together with descent maps $d^{(n)}:B^{(n+1)}\rightarrow\underline{M}^{(n)}$.

A <u>finite sequence of generations</u> is a sequence $\underline{G}^{(n)}$, $\underline{G}^{(n+1)}$, ... $\underline{G}^{(n+k)}$ of generations together with configurations $\underline{C}^{(n)}$, $\underline{C}^{(n+1)}$, ... $\underline{C}^{(n+k)}$ and together with descent maps $d^{(k)}:B^{(k+1)}\rightarrow\underline{M}^{(k)}$ $n<k<n-1$.

If $a\epsilon\underline{B}^{(n+1)}$ and $M\epsilon\underline{M}^{(n)}$ and $d(a) = M$ then denote the map element in d from a to M as "d_{aM}."

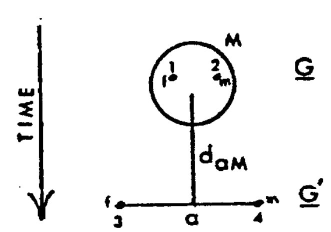
Notice that while there must be a descent map element from each element a of $\underline{B}'$ to some distinct element M of $\underline{M}$, there may be elements of $\underline{M}$ which have no descent map elements to members of $\underline{B}'$.

Notational Convenience:

"Let $\underline{G}^{(n)}$ be a generation in a sequence of generations" will be used to mean: "Let ... , $\underline{G}^{(n)}$, $\underline{G}^{(n+1)}$, ... be an (infinite) sequence of generations with configurations ... , $\underline{C}^{(n)}$, $\underline{C}^{(n+1)}$, ... , and descent maps ... , $d^{(n)}:B^{(n+1)}\rightarrow\underline{M}^{(n)}$, ... , for all n and let $\underline{G}^{(n)}$ be a generation in this sequence."

Note that under certain circumstances, the existence of a finite sequence permits construction of an infinite sequence of generations. For this reason, "sequence of generations" will mean "infinite sequence of generations" unless otherwise stated. A theorem on cyclic sequence construction is presented later.

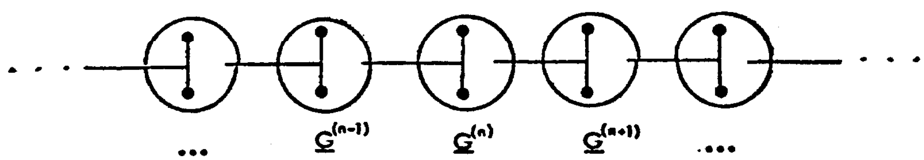
Notation is developed for displaying these pairs ($\underline{C}$,$\underline{C}'$) in a graphic manner. Use a dot "." to represent individuals $\epsilon\underline{G}$ and use a labeled dot ".m" of ".f" to show that the individual p represented by the dot is either pϵf or pϵf, respectively. A line between dots (labeled or unlabeled dots) will show equivalence mod $\underline{B}$ for individuals in $\underline{G}$, and mod $\underline{B}'$ for individuals in $\underline{G}'$. A circle around two dots show equivalence mod $\underline{M}$ or mod $\underline{M}'$, respectively. Lines and circles may be labeled. On occasion a line will be allowed to branch, to include more than two individuals mod $\underline{B}$, etc. A line with no dot at the end, from inside a circle to a line with dots, or to a single dot, shows a particular descent map element.



In this diagram, the pictorial representation depicts the following sets: $\underline{G}$ = {1,2} ; $\underline{B}$ = {(1),(2)} ; $\underline{G}$ = {1,2} ; $\underline{B}$ = {(1)(2)} ; $\underline{M}$ = {M} = {{1,2}} ; $\underline{S}$ = m f = {2} {1} ; $\underline{G}'$ = {3,4} ; $\underline{B}'$ = {a} = {{3,4}} ; $\underline{M}'$ = Ø ; $\underline{S}'$ = m f = {4} {3} ; d = {d_{aM}}. Thus, $\underline{G}'$ is a generation with configuration $\underline{C}'$ = {$\underline{M}'$,$\underline{B}'$,$\underline{S}'$}. The elements of $\underline{M}'$, $\underline{B}'$ and $\underline{S}'$ are enumerated above. Individuals 1 and 2 of $\underline{G}'$ are the ancestors of 3 and 4 in $\underline{G}$, which has configuration C = {$\underline{M}$,$\underline{B}$,$\underline{S}$} whose elements are also enumerated above. The descent may d:B'→M consists of the single element d_{aM}.

In addition, an arrow with two shafts will be used between drawings which show marriage structure on a generation which previously had no indicated marriages, while a single shafted arrow will be used to show application of descent maps from an existing generation to a "new" generation. (See theorems D and F' for examples.)

Note that because of definitions 1 and 2, every generation in a sequence of generations has at least the following minimal statistics: $|\underline{M}^{(n)}|\geq 1$, $|\underline{B}^{(n)}|\geq 1$, $|\underline{G}^{(n)}|\geq 2$, $|m^{(n)}|\geq 1$, $|f^{(n)}|\geq 1$, for all n. If a particular sequence had only these statistics, it could be illustrated as follows:



DEFINITION 3: If $\underline{G}$ and $\underline{G}'$ are a pair of generations with configurations $\underline{C}$ and $\underline{C}'$ respectively, and d:$\underline{B}'$→$\underline{M}$ is a descent map from $\underline{B}'$ to $\underline{M}$, and $d_{aM}\epsilon d$, and pϵe and qϵm, then q is a <u>parent</u> of p, and p is an <u>offspring</u> of q.

Let $\underline{G}^{(n)}$ be a generation in a (finite or infinite) sequence of generations. Let $p\epsilon\underline{G}^{(n)}$ and suppose $\underline{G}^{(m)}$ is another generation in this sequence and m<n, and $q\epsilon\underline{G}^{(n)}$. A <u>descent chain</u> from q to p exists if there exists a sequence $q^{(m)}$, $q^{(m+1)}$, ... , $q^{(n)}$ such that $q^{(n)}\epsilon\underline{G}^{(n)}$ is an offspring of $q^{(i-1)}\epsilon\underline{G}^{(i-2)}$, $q^{(i-2)}$ is a parent of $q^{(i-1)}$, ... , $q^{(m)}$ is a parent of $q^{(m+1)}$, and p = $q^{(n)}$ and q = $q^{(m)}$.

If there exists a descent chain from q to p, then q is an <u>ancestor</u> of p, and p is a <u>descendant</u> of q.

Note that if q is a parent of p, then q is also an ancestor of p; if p is an offspring of q, then p is also a descendant of q. Note also that for a pair of generations ($\underline{G}$,$\underline{G}'$) with configurations ($\underline{C}$,$\underline{C}'$) and a descent map d:$\underline{B}'$→$\underline{M}$, if d(a) = M, then there exist, for each pϵa, to distinct elements of qϵM, each of which is an ancestor of p.

DEFINITION 4: Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations, and suppose $a\varepsilon\underline{B}^{(n)}$ and $b\varepsilon\underline{B}^{(n)}$ and $p\varepsilon a$ and $q\varepsilon b$. Let $\underline{G}^{(m)}$ be some other generation in the sequence such that $m<n$, and let $j = n-m$. The p and q are <u>j-removed</u> iff there exists $p'\varepsilon\underline{G}^{(m)}$ such that both p and q are each descended from p'.

If p and q are j-removed, write $R^{j}(p,q)$.

Note: $p\cong$ mod $\underline{B}$ iff $R^{1}(p,q)$.

Definition 4 allows discussion of "descent relations" that extend "backward" over several generations, while def. 5 below allows discussion of affine relations which extend "across" a single generation.

DEFINITION 5: Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations, and suppose $p\varepsilon\underline{G}^{(n)}$ and $q\varepsilon\underline{G}^{(n)}$ and $p\varepsilon a\varepsilon\underline{B}^{(n)}$ and $q\varepsilon b\varepsilon\underline{B}^{(n)}$ and $d(a) = M_a\varepsilon\underline{G}^{(n-1)}$ and $d(b) = M_b\varepsilon\underline{G}^{(n-1)}$.

If $M_a = M_b$ then $a = b$ and $p\cong$ mod $\underline{B}^{(n)}$. Call this condition "p and q are <u>o-linked</u>," written $L^{o}(p,q)$.

If $M_a \neq M_b$ then $a \neq b$. Suppose $p'\varepsilon Ma$ and $q'\varepsilon Mb$ are ancestors of p and q respectively, and $q'\cong p'$ mod $\underline{B}^{(n)}$. Then p and q are <u>1-linked</u>, or first linked, written $L^{1}(p,q)$. If $p'\not\cong q'$ mod $\underline{B}^{(n)}$ but there exist $p''\cong p'$ mod $\underline{B}^{(n)}$ and $q''\cong q'$ mod $\underline{B}^{(n)}$ and $p''\cong q''$mod $\underline{B}^{(n)}$ then p and q are <u>2-linked</u>.

Similarly, if p and q are not i-linked for all $i<j$, but a series of j+1 distinct B-equivalence classes linked by j distinct M-equivalence classes, then p and q are <u>j-linked</u>, written $L^{j}(p,q)$.

The particular sequence of B-sets linked through M-sets of $\underline{G}^{(n-1)}$ from p' to q', inclusive, forms a <u>chain</u> of j+1 B-sets, and a <u>chain</u> of j M-sets.

Remark: the following are easily proved:

(i) simply extended removal: Suppose G is a generation in s sequence of generations, and assume p and q $\underline{G}^{(n)}$ and $P^{j}(p,q)$. then $R^{j+n}(p,q)$ for an positive integer n.

(ii) elementary extinction: Every generation in an infinite sequence contains at least one pair of individuals (p,q) such that $R^{1}(p,q)$.

Motivation for the next definition should be apparent, except to note that the distinction of several types of isomorphism is necessitated by the structure imposed by definition 1.

DEFINITION 6: Suppose $\underline{G}$ and $\underline{G}'$ are two generations with configurations $\underline{C}$ and $\underline{C}'$ respectively, and suppose there exists a 1-1 onto map $f:\underline{G}\to\underline{G}'$.

Then G and $\underline{C}'$ are B-isomorphic iff for $p\in\underline{G}$ and $q\in\underline{G}$, if $p\equiv q$ mod $\underline{B}$ then $f(p)\equiv(f)q$ mod $\underline{B}'$.

$\underline{C}$ and $\underline{C}'$ are M-isomorphic iff for $p\in\underline{G}$ and $q\in\underline{G}$ if $p\equiv q$ mod $\underline{M}$ then $f(p)\equiv(f)q$ mod $\underline{M}'$.

$\underline{C}$ and $\underline{C}'$ are S-isomorphic iff for $p\in\underline{G}$ and $q\in\underline{G}$ if $p\equiv q$ mod $\underline{S}$ then $f(p)\equiv(f)q$ mod $\underline{S}'$.

If $\underline{C}$ and $\underline{C}'$ are B-, M- and S- isomorphic, then they are completely isomorphic, or simply isomorphic.

The sign "$\equiv$" will be used between isomorphic configurations (completely isomorphic configurations), the signs "$\underset{=}{B}$" , "$\underset{=}{M}$" , and "$\underset{=}{S}$" will be used between B-, M- and S-isomorphic configurations, respectively. If $\underline{G} \equiv G'$, then $\underline{G}' \equiv \underline{G}$. If $\underline{G} \equiv \underline{G}' \equiv \underline{G}''$ then $\underline{G}'' = \underline{G}$. If $\underline{G} \underset{=}{B} G'$ then $\underline{G}' \underset{=}{B} G$, and if $\underline{G}\overset{B}{=}\underline{G}'\overset{B}{=}\underline{G}''$ then $\underline{G}\overset{B}{=}\underline{G}''$, and similarly for S- and M- isomorphisms.

DEFINITION 7: Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations, and $p\in\underline{G}^{(n)}$ and $q\in\underline{G}^{(n)}$ and $p\equiv q$ mod $\underline{B}^{(n)}$. Let p' and q' be ancestors in $\underline{G}^{(n-1)}$ of p and q respectively.

Then if $p'\equiv q'$ mod $\underline{S}^{(n-1)}$ then p and q are parallel linked. If $p'\not\equiv q'$ mod $\underline{S}^{(n-1)}$ then p and are cross linked. Notate this "cross $L^1(p,q)$" and "parallel $L^1(p,q)$" respectively.

DEFINITION 8: Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations. A marriage rule A#, where "#" is a number, is satisfied in the sequence $\{\underline{G}^{(n)}\}$ (or, the sequence $\{\underline{G}^{(n)}\}$ satisfies A) if whenever $p,q\in\underline{G}^{(n)}$ for all $\underline{G}^{(n)}\in\{\underline{G}^{(n)}\}$ and $p\equiv q$ mod $\underline{M}^{(n)}$, then A# is true, provided A# is a conjunction of statements drawn from this list:

(1) for some positive integer i, $R^i(p,q)$
(2) for some positive integer i, not $R^i(p,q)$
(3) for some positive integer i, $L^i(p,q)$
(4) for some positive integer i, not $L^i(p,q)$
(5) there exist ancestors p' and q' of p and q respectively in $\underline{G}^{(n-i)}$, for ositive integer i, cross $L^1(p',q')$
(6) there exist ancestors p' and q' of p and q respectively in $\underline{G}^{(n-i)}$, for ositive integer i, not cross $L^1(p',q')$
(7) there exist ancestors p' and q' of p and q respectively in $\underline{G}^{(n-i)}$, for ositive integer i, parallel $L^1(p',q')$
(8) there exist ancestors p' and q' of p and q respectively in $\underline{G}^{(n-i)}$, for some positive integer i, not parallel $L^1(p',q')$

A variety of marriage rules with possible English language interpretations are presented below. Note that while the rules are unique, the interpretations may not be unique.

Rule Name	Rule	Interpretation
A1	None	"Random," sexual mating
A2	(2)1	"Sibling exclusion"
A3	(2)1, (6)1	"Cross cousin exclusion"
A4	(2)1, (8)1	"Parallel cousin exclusion"
A5	(2)1, (4)1	"First cousin exclusion"
A6	(2)1, (3)2	"Four section system," also "double first cousins"

"Contradictory" marriage rules which require both $R^i(p,q)$ and not $R^i(p,q)$, or which require both $L^i(p,q)$ and not $L^i(p,q)$ for the same value of values of i are clearly not satisfied by any sequence of generations.

DEFINITION 9: A stable configuration of period k, of simply a k-stable configuration, is a configuration $\underline{C}^{(n)}$ such that there exists a finite sequence $\underline{C}^{(n)}, \ldots, \underline{C}^{(n+k)}$ and $\underline{C}^{(n)} = \underline{C}^{(n+k)}$, and such that if $\underline{C}'^{(n)}, \ldots, \underline{C}'^{(n+i)}$ is another sequence with $\underline{C}'^{(n)} = \underline{C}'^{(n+i)}$, then $k<i$.

Theorem: (cyclic sequence construction) Suppose a sequence of generations is k-stable, such that $\underline{C}^{(n)} = \underline{C}^{(n+k)}$. Then there exists an infinite sequence $\ldots, \underline{C}_v', \ldots, \underline{C}_v^{(k)}, \underline{C}_v^{(k+1)}, \ldots$ such that $\underline{C}_v'^{(i)}, \underline{C}_v'^{(i+k)}$ for all i.

Proof: Let $\underline{C}^{(n)}, \underline{C}^{(n+1)}, \ldots, \underline{C}^{(n+k)}$ be a stable sequence, with descent maps $d^{(i)}: \underline{B}^{(i+1)} \to \underline{M}^{i}$, $n \leq i \leq n+k-1$. Construct a new sequence as follows ($\underline{C}_v^{(1)} = \underline{C}^{(n)}$, $\underline{C}_v^{(2)} = \underline{C}^{(n+1)}, \ldots, \underline{C}_v^{(k)} = \underline{C}^{(n+k-1)}$, $\underline{C}_v^{(k+1)} = \underline{C}^{(n+k)} = \underline{C}_v^{(1)}$, $\underline{C}_v^{(k+2)} = \underline{C}_v^{(2)}, \ldots$ and $d_v^{(1)} = d^{(n)}$, $d_v^{(2)} = d^{(2)}, \ldots, d_v^{(k)} = d^{(n+k-1)}$, $d_v^{(k)} = d_v^{(1)}, \ldots$

The new sequence $\{\underline{C}_v^{(n)}\}$ is thus cyclic in n mod k, and the descent relations $d_v^{(n)}$ are also cyclic in n mod k.

Q.E.D.

III. Developments

THEOREM A: Let $\ldots, \underline{G}^{(n)}, \ldots$ be a sequence which satisfies A2. then for all n, $|\underline{G}^{(n-1)}| \geq 4$, $|\underline{B}^{(n-1)}| \geq 2$, $|\underline{M}^{(n-1)}| \geq 2$, $|m^{(n-1)}| \geq 2$, $|f^{(n-1)}| \geq 2$.

Proof: Let p and q be elements of $\underline{G}^{(n)}$ such that $p \equiv q \bmod \underline{M}^{(n)}$. Let $a \varepsilon \underline{B}^{(n)}$ and $b \varepsilon \underline{B}^{(n)}$ $p \varepsilon a$ and $q \varepsilon b$. By hypothesis (A2) $d(a) \neq d(b)$, hence $a \neq b$.

Therefore $|\underline{M}^{(n-1)}|\geq 2$, $|\underline{G}^{(n-1)}|\geq 4$, $|\underline{B}^{(n-1)}|\geq 2$ since each individual in each M-set must be from different B-sets, and $|m^{(n-1)}|\geq 2$, $|f^{(n-1)}|\geq 2$ from definition 1.

Q.E.D.

Theorems B, B' and C follow immediately from theorem A.

THEOREM B: Let ... $G^{(n)}$, ... be a sequence of generations which satisfies A6. Then: $|\underline{G}^{(n)}|\geq 4$, $|\underline{B}^{(n)}|\geq 2$, $|\underline{M}^{(n)}|\geq 2$ for all n.

THEOREM B': Let ... $\underline{G}^{(n)}$, ... be a generation in a sequence of generations which satisfies A6, then for each n there exists at least one p' ϵ $\underline{G}^{(n-1)}$ and one q'$\epsilon\underline{G}^{(n-1)}$, p' $\neq$ q', such that L^1(p',q').

<u>Proof</u>: in the proof of theorem A, let p'ϵM' be an ancestor of p and q'ϵM'' be an ancestor of q. Note that p'=q' is the prohibited by theorem A' and by the prohibition of R^1(p,q) by A6.

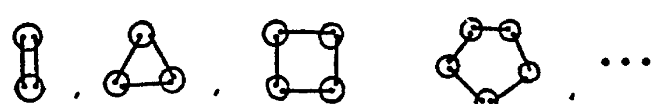
Q.E.D.

THEOREM C: Let ... , $\underline{G}^{(n)}$, ... be a sequence of generations which satisfies A2. Then $|\underline{G}^{(n)}|\geq 4$, $|\underline{B}^{(n)}|\geq 2$, $|\underline{M}^{(n)}|\geq 2$ for all n.

Note that while theorems A, B, and C specify conditions which must at least be true for viable sequences under marriage rules, they do not guarantee that any structure exists which satisfies these conditions, nor do they specify particular structures.

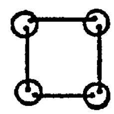
DEFINITION 10: All those configurations which are isomorphic to one another form an <u>isomorphism class</u>.

A <u>configurational element</u> is one of the following isomorphism classes:



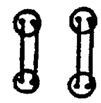
$|\underline{M}|=2$, $|\underline{M}|=3$, $|\underline{M}|=4$, $|\underline{M}|=5$, ...

Notice that an isomorphism class can be unambiguously represented up to an isomorphism of the S-labels, by display of a member of the class. Note also that in the configurations below, both have $|\underline{M}| = 4$, but one consists of a single configurational element, while the other consists of the union of two configurational elements each of which has $|\underline{M}| = 2$. If an isomorphism class



M4

FIGURE I



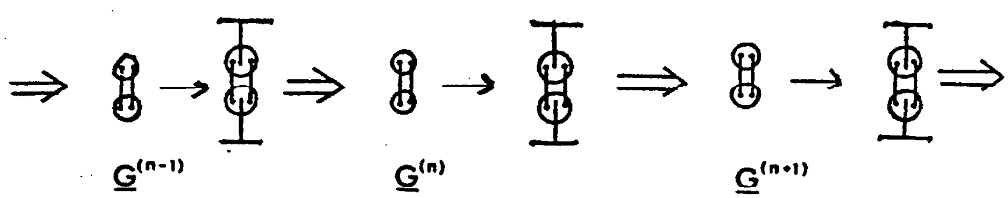
2(M2)

FIGURE II

is named "M#" where # is the number of M-sets in the class, then figure I above consists of the single configurational element M4, while figure II is the union of two M2 isomorphism classes. This may unambiguously be notated "2(M2)," describing a single generation which $\underline{G}$ whose configuration $\underline{C} = (\underline{M},\underline{B},\underline{S})$ is the union of two separate configurations $\underline{C}_1(\underline{M}_1,\underline{B}_1,\underline{S}_1)$, $C_2 = (\underline{M}_2,\underline{B}_2,\underline{S}_2)$, i.e., $\underline{M} = \underline{M}_1 U \underline{M}_2$, $\underline{B} = \underline{B}_1 U \underline{B}_2$, $f = f_1 U f_2$, $m = m_1 U m_2$.

Sequences may be uniquely named if all configurations are isomorphic to a given configuration. For example, a sequence which has $\underline{C}^{(n)} = M2$ for all n may be called $(n)\{\underline{C}^{(n)} = M2\}$. This is suggestive of similar notations for cyclic repetitions of configurations, but these will not be exploited here.

THEOREM D: The sequence of $(n)\{\underline{C}^{(n)} = M2\}$ which has $\underline{C}^{(n)} = M2$ for all n



satisfies theorem C with equalities in place of inequalities, and any other sequence which thus satisfies theorem C is isomorphic to $(n)\{\underline{C}^{(n)} = M2\}$.

Proof: By inspection, $(n)\{\underline{C}^{(n)} = M2\}$ has $|\underline{M}^{(n)}| = 4$, $|\underline{B}^{(n)}| = 2$, $|\underline{G}^{(n)} = 4$ for all n.

To show all other sequences with this property must be isomorphic to $(n)\{\underline{C}^{(n)} = M2\}$, consider the two examples below:

CLASS I 

CLASS II 

Since each M-class by definition 1 must have $|M| = 2$, these are the only two configurations other than M2 which have $|\underline{M}| = 2$ and also have $|\underline{G}| = 4$. Class I has $|\underline{B}| = 4$, while class II has $|\underline{B}| = 3$. Since M2 has $|\underline{B}| = 2$, the theorem is shown.

Q.E.D.

Note that because of definition 5, theorem D implies theorem D'.

THEOREM D': The sequence $(n)\{\underline{C}^{(n)} = M2\}$ satisfies A6, and satisfies theorem B' with equalities in place of inequalities, and any other sequence satisfying theorem B' with equalities in place of inequalities is isomorphic to $(n)\{\underline{C}^{(n)} = M2\}$.

THEOREM E: Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations which satisfies A5. Then $|\underline{B}^{(n-1)}| \geq 4$ for all n.

<u>Proof</u>: Let $p,q \in \underline{G}^{(n)}$. Then not $R^1(p,q)$ implies distinct descent chains, from ancestors in $\underline{G}^{(n-1)}$, hence $|\underline{M}^{(n-1)}| > 2$. Let p' and q' in $\underline{G}^{(n-1)}$ be ancestors or p or q respectively. Then not $L^1(p,q)$ implies $p' \neq q'$ mod $\underline{B}^{(n-1)}$. Since each M-set in $M^{(n-1)}$ has 2 individuals, and none are equivalent mod $\underline{B}^{(n-1)}$, $|\underline{B}^{(n-1)}| \geq 4$.

THEOREM E': Suppose $\underline{G}^{(n)}$ is a generation in a sequence of generations which satisfies A5. Then $|\underline{G}^{(n)}| \geq 8$, $|\underline{B}^{(n)}| \geq 4$, $|\underline{M}^{(n)}| \geq 4$ for all n.

<u>Proof</u>: Apply theorem E to $\underline{G}^{(n)}$ and to $\underline{G}^{(n+1)}$, finding respectively that $|\underline{B}^{(n-1)}| \geq 4$ and $|\underline{B}^{(n)}| \geq 4$. Since $|\underline{B}^{(n-1)}| \geq 4$ and descent maps are 1-1, $|\underline{M}^{(n)}| \geq 4$ (since $|\underline{M}^{(n)}| \geq |\underline{B}^{(n-1)}|$). And since $|\underline{G}^{(n)}| \geq 2|\underline{M}^{(n)}|$, $|\underline{G}^{(n)}| \geq 8$.

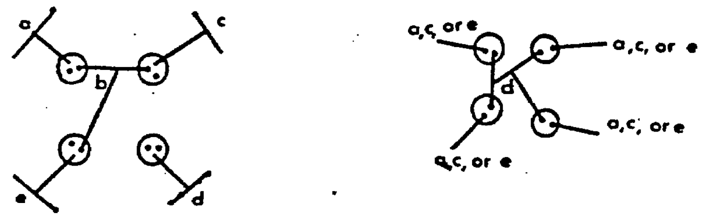
Q.E.D.

THEOREM F: There exists a sequence of generations which satisfies A5, satisfies theorem E' with equalities in place of inequalities, and is k-stable for k = 1.

<u>Proof</u>: The sequence is $(n)\{\underline{C}^{(n)} = 2(M2)\}$.

THEOREM F': (equipartition from A5) Any configuration $\underline{C}^{(n)}$ in a sequence which satisfies A5 and theorem E' with equalities in place of inequalities must have $|b_i| = 2$ for all $b_i \epsilon \underline{B}^{(n)}$, and for all n.

Proof: To show equipartition, note that if there is one B-set b with $|b| = 3$ in a generation. Then three of the M-sets will have 1-linked elements in the next generation. Because of $L^1(p,q)$ prohibition, these three can only marry from the fourth, but in the succeeding generation, all will be 1-linked (illustrated below).



The same illustration shows the consequences of $|b| = 4$ are that L^1-prohibition prevents all marriages.

Note that $|b| = 5$ is prohibited, since 8-5 = 3, hence only 3 M-sets at most may be completed, in violation of $|\underline{M}^{(n)}| = 4$. Similarly for b-sets of size 6, 7, or 8, while if any B-set is size 1, then some other B-set is size ≥ 3; ($|\underline{B}| = 0$ is meaningless).

Q.E.D.

DEFINITION 11: Let A be a k-stable marriage rule on a finite sequence of generations which satisfies A. Then $\mathcal{C}$ is a representation of A is $\mathcal{C} = \underline{C}^{(n)}, \ldots, \underline{C}^{(n+k)}$ and $\underline{C}^{(n)} = \underline{C}^{(n+k)}$. Each particular sequence which constitutes a representation has an initial configuration $\underline{C}^{(n)}$ and a final configuration $\underline{C}^{(n+k)}$, each configuration in a representation is called a step in the representation.

A representation therefore has k+1 steps, including the initial and final steps.

DEFINITION 12: Let A and A' be availability rules (not necessarily different) with representations $\mathcal{C}$ and $\mathcal{C}'$ respectively. Then $\mathcal{C} = \mathcal{C}'$ iff k = k', and for each step in $\mathcal{C}$ and $\mathcal{C}'$, $\underline{C}^{(i)} = \underline{C}^{(i)}$.

Therefore there may possibly be as many as k non-isomorphic representations of any particular marriage rule, once it is known that a particular k-stable sequence exists which satisfies the rule. In addition, there may be many possibly distinct k-stable sequences, each larger than the previous in the size of its generations, or of the generation of some particular step. This suggests the importance of the following definition.

DEFINITION 13: Let A be a marriage rule and let $\mathcal{C}, \mathcal{C}', \mathcal{C}'', \ldots$ be a representation of A. Call a representation $\mathcal{C} = \underline{C}^{(1)}, \underline{C}^{(2)}, \ldots, \underline{C}^{(k)}$ a minimal representation of

A iff for any other representation θ' of A, $\theta' = \underline{C}'^{(1)}, \underline{C}'^{(2)}, \ldots, \underline{C}'^{(k)}$, and for all $1 \le i \le k$, $|\underline{B}^{(i)}| \le |\underline{B}'^{(i)}|$, $|\underline{M}^{(i)}| \le |\underline{M}'^{(i)}|$, $|\underline{G}^{(i)}| \le |\underline{G}'^{(i)}|$.

Call the minimal numbers $|\underline{B}^{(i)}| = \beta_i$, $|\underline{M}^{(i)}| = \mu_i$, and $|\underline{G}^{(i)}| = \gamma_i$ thus discovered structural numbers of A.

Note: Inferences from later generations to earlier ones in a sequence can only be made through ancestors, which are members of M-sets, or to individuals linked to ancestors, which are also members of M-sets. Therefore all individuals in a step of a minimal representation are members of M-sets.

DEFINITION 14: The state of a generation $\underline{G}$, if $\underline{C}$ is isomorphic to any (set of) configurational element(s) is the (set of) configurational element(s) isomorphic to the configuration $\underline{C}$ on $\underline{G}$. An orbit is a sequence of states.

Thus not every generation has a defined state, but for example, the minimal representation of A5 is a sequence of generations of length 2, since k = 1, and each step in the representation is in the state 2(M2), while the orbit of A5 is simply a sequence of 2(M2) states. One advantage gained by talking in terms of states rather than generations is that states may repeat themselves, while each generation is unique.

The significant structural questions are these: the existence and discovery of a minimal sequence for a given marriage rule A; the uniqueness of non-uniqueness of the orbit of states (and the existence of an orbit of states) for a given rule A; the existence and uniqueness of structural numbers for each rule.

THEOREM G: Suppose A is a k-stable marriage rule with minimal representation θ. If $\underline{C}^{(n)}$ is a step in θ and $|\underline{B}^{(n)}| = \beta_n$, $|\underline{M}^{(n)}| = \mu_n$, and $|\underline{G}^{(n)}| = \gamma_n$ then there exist numbers β, μ, and γ where γ is even, such that for all n, $\beta = \beta n$, $\gamma = \gamma n$, $\mu = \mu n$ and $|m_n| = |f_n| = \mu = 1/2\ \gamma$ and $1 \le \beta \le \mu$.

Proof: by cyclic sequence construction concatonate k-stable sub-sequences such that the final step of one sub-sequence is the initial step of the next sub-sequence. Each sub-sequence is isomorphic to all others, and clearly each distinct isomorphic configuration, and thus each set of structural numbers, recurs with period k in the constructed sequence.

(i) Suppose that at some step i, A implies that at some preceeding step (i-j) that $|\underline{G}^{(i-j)}| = \gamma_{i-j}$ and that for all n $0 \le n \le k$, $\gamma_{i-j} \ge \gamma_{i-n}$. Since the definition of A requires that the same conditions apply to each step, application of A to $\underline{G}^{(i-n+j)}$ implies that $\gamma_{i-n} = \gamma_{i-j}$. Therefore there is a unique $\gamma = \gamma i$ for all i in θ.

(ii) By similar arguments there are unique $\beta = \beta i$ and $\mu = \mu i$.

(iii) Clearly $|m_i| \geq \mu$ and $|f_i| \geq \mu$ for all i in θ, by definition 1. Likewise there are unique $|m_i| = \bar{m}$ and $|f_i| = \bar{f}$ by the same arguments as above. And since $\bar{m}+\bar{f} = \gamma$ and $\bar{m} = \bar{f} = \mu$ then γ is even, and $\bar{m} = \bar{f} = 1/2\ \gamma$.

(iv) If μ is unique, then because descent relations are 1-1 onto B-sets $\beta < \mu$. And by definition 1, $\beta \geq 1$.

Q.E.D.

CONJECTURE G': $\beta = \mu$ in every minimal representation.

CONJECTURE G'': Every step in a minimal representation is partitioned into β equal B-sets of size 2.

Concluding Comments:

Minimal representation are clearly efficient structures for depicting the operations of marriage rules. The theory of minimal structures should make it possible to study a large class of marriage rules in a rigorous way. And while study of abstract properties of marriage rules in the present format will hopefully have mathematical interest of its own, possible areas of application should not be neglected. For example, the structural numbers of a marriage system are predictive of its demographic statistics. Also, there are obvious extensions of the formalism to labeled maps or to label transferring maps, which would produce minimal sequences which are mathematical groups. Such topics are also reserved for separate treatment.

BIBLIOGRAPHY TO APPENDIX I

Cook, E. A.
1969 "Marriage Among the Manga," Pigs, Pearlshells and Women, Edited by Glasse and Meggitt, Prentice-Hall, Englewood Cliffs, N.J., pp. 96-116.

Duchamp, T.
1972 "Matrix Methods in the Theory of Marriage Networks," Unpublished paper.

Levi-Strauss
1969 Elementary Structures of Kinship, Beacon Press, Boston, MA.

Livingstone, F. B.
1959 "A Formal Analysis of Prescriptive Marriage Systems Among the Australian Aborigines," in Southwestern Journal of Anthropology, Vol. 15, pp. 361-372.

Ruheman, B.
1945 "A Method for Analyzing Classificatory Relationship Systems," in Southwestern Journal of Anthropology, Vol. 1, pp. 531-576.

Wright, S.
1921 "Systems of Mating II: Inbreeding and the Composition of a Population," in Genetics 6:124-143, March.

Zuidema, T.
1965 "American Social Systems and Their Mutual Similarity," Offprint 31, Institute of Latin American Studies, University of Texas.