Selected Portions of:

A CLASSIFICATION OF MAPPINGS BETWEEN FINITE SETS AND SOME APPLICATIONS

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Let A and B be non-empty finite sets,

$$A = \{a_1, a_2, \ldots, a_n\}, \qquad B = \{b_1, b_2, \ldots, b_n\}.$$

Let B^A denote the set of all mappings μ from A to B,

$$B^A = \{\mu \mid \mu : A \longrightarrow B\}.$$

Then the cardinality of B^A is

$$\text{card } B^A = (\text{card } B)^{\text{card } A} = m^n.$$

The elements of B^A shall be classified by three different equivalence relations which, by Greek numerals, are called proto-equivalence, deutero-equivalence, and trito-equivalence (protos = first; deuteros = second; tritos = third; compare proton ^1_1H, deuteron ^2_1H, triton ^3_1H in nuclear physics). These terms were used first by G. Günther [3] for a classification of the functions of a many-valued propositional calculus.

Let proto-equivalence be denoted by $\overset{p}{\sim}$. The proto-equivalence of two mappings μ_1 and μ_2 of B^A is given by

Definition 1.

$$\mu_1 \overset{p}{\sim} \mu_2 \iff \text{card } {}^{A}/_{\ker \mu_1} = \text{card } {}^{A}/_{\ker \mu_2}$$

where card ${}^{A}/_{\ker \mu}$ is the cardinality of the quotient set ${}^{A}/_{\ker \mu}$ of A by the kernel of μ. The kernel of μ, denoted by ker μ or $\mu^{-1}\mu$, is an equivalence relation in A and hence a subset of the product set $A \times A$. For $x, y \in A$:

$$(x,y) \in \ker \mu \text{ if and only if } \mu(x) = \mu(y).$$

It is easy to see that the proto-equivalence is reflective, symmetric, and transitive. Hence B^A is partitioned into pairwise disjoint non-empty subsets. The number of these subsets, i.e., the number of proto-equivalence classes is given by

Theorem 1.

$$\text{card } {}^{B^A}/_{\overset{p}{\sim}} = \min \{\text{card } A, \text{card } B\}$$

where ${}^{B^A}/_{\overset{p}{\sim}}$ is the quotient set of B^A by the proto-equivalence $\overset{p}{\sim}$.

Proof. Let $\mu : A \longrightarrow B$ be an element of B^A and let be

$M = \min\{\text{card } A, \text{ card } B\}$.

The number of equivalence classes in $A/_{\ker \mu}$ cannot be greater than M, but it can take all integer values between 1 and M because for every partition of A into k disjoint subsets where $1 \leq k \leq M$, there exists a mapping in B^A whose kernel yields this partition. Hence there are M proto-equivalence classes of mappings, i.e.,

$$\text{card } B^A/_{\underset{\sim}{p}} = \min\{\text{card } A, \text{ card } B\} .$$

Let deutero-equivalence be denoted by $\overset{d}{\sim}$.

Definition 2.

$$\mu_1 \overset{d}{\sim} \mu_2 \iff A/_{\ker \mu_1} \cong A/_{\ker \mu_2}$$

where the isomorphism between $A/_{\ker \mu_1}$ and $A/_{\ker \mu_2}$ is defined by

$A/_{\ker \mu_1} \cong A/_{\ker \mu_2} \iff$ there exists a one-to one correspondence $: A/_{\ker \mu_1} \longrightarrow A/_{\ker \mu_2}$ such that card $([a_i]_{\ker \mu_1})$ = card $[a_i]_{\ker \mu_1}$ for all $a_i \in A$.

$[a_i]_{\ker \mu}$ is the equivalence class of a_i relative to the kernel of μ;

$$[a_i]_{\ker \mu} = \{a \in A | (a_i, a) \in \ker \mu\} .$$

Obviously the deutero-equivalence is reflective, symmetric, and transitive. Thus B^A is partitioned into disjoint subsets. The number of these subsets is given by

Theorem 2.

$$\text{card } B^A/_{\underset{\sim}{d}} = \sum_{k=1}^{M} P(n,k)$$

where $M = \min\{\text{card } A, \text{ card } B\}$, n = card A, and P(n,k) = number of partitions of n into k integer summands without regard to order.

Proof. Definition 2 implies that two mappings are deutero-equivalent if and only if their kernels yield the same partition of n without regard to order. There is a deutero-equivalence class of mappings for each partition of n into k integer summands where $1 \leq k \leq M$; hence

$$\text{card } B^A/_{\underset{\sim}{d}} = \sum_{k-1}^{M} P(n,k) .$$

Corollary 1. The number P(n,k) of partitions of an integer $n \geq 1$ into k integer parts without regard to order is given by the recurrence relation

$$P(n,k) = P(n-1,k-1) + P(n-k,k)$$

with boundary values

$P(n,1) = 1$; $P(n,n) = 1$; $P(n,k) = 0$ for $k > n$.

Proof. It is

$$n = k + (n - k) = \underbrace{1 + 1 + \ldots + 1}_{k\text{-times}} + (n - k) .$$

$(n - k)$ can be added to the last 1 in $P(n-k,1)$ ways, or to the last two 1's in $P(n-k,2)$ ways, or to the last three 1's in $P(n-k,3)$ ways, etc. Therefore,

$$P(n,k) = P(n-k,1) + P(n-k,2) + \ldots + P(n-k,k-1) + P(n-k,k).$$

This implies

$$P(n-1,k-1) = P(n-k,1) + P(n-k,2) + \ldots + P(n-k,k-1) .$$

Therefore,

$$P(n,k) = P(n-1,k-1) + P(n-k,k) .$$

The boundary values are obvious.

Tables of the number $P(n,k)$ are published in [5] (where $P(n,k) \equiv P(n,m) - p(n-m,m)$). A short table is given on the following page.

Let trito-equivalence be denoted by $\overset{t}{\sim}$.

Definition 3.

$$\mu_1 \overset{t}{\sim} \mu_2 \Longleftrightarrow {}^{A}/_{\ker \mu_1} = {}^{A}/_{\ker \mu_2} .$$

That means, $[a_i]_{\ker \mu_1} = [a_i]_{\ker \mu_2}$ for all $a_i \in A$.

Evidently the trito-equivalence is reflective, symmetric, and transitive. Therefore, B^A is partitioned into disjoint subsets. The number of these subsets is given by

Theorem 3.

$$\text{card } {}^{B^A}/_{\overset{t}{\sim}} = \sum_{k=1}^{M} S(n,k)$$

where $M = \min\{\text{card } A, \text{card } B\}$, $n = \text{card } A$, and $S(n,k)$ = number of ways of partitioning a set of n elements into k non-empty subsets.

Proof. Definition 3 implies that two mappings are trito-equivalent if and only if their kernels yield the same partition of the set A into non-empty subsets. There

Partition Numbers P(n,k)

n \ k	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1												
2	1	1											
3	1	1	1										
4	1	2	1	1									
5	1	2	2	1	1								
6	1	3	3	2	1	1							
7	1	3	4	3	2	1	1						
8	1	4	5	5	3	2	1	1					
9	1	4	7	6	5	3	2	1	1				
10	1	5	8	9	7	5	3	2	1	1			
11	1	5	10	11	10	7	5	3	2	1	1		
12	1	6	12	15	13	11	7	5	3	2	1	1	
13	1	6	14	18	18	14	11	7	5	3	2	1	1

is a trito-equivalence class of mappings for each partition of A into k non-empty subsets where $1 \leq k \leq M$; hence

$$\text{card } B^A/_{\underset{\sim}{t}} = \sum_{k=1}^{M} S(n,k) \ .$$

Corollary 2. The number $S(n,k)$ of ways of partitioning a set of n elements into k non-empty subsets is equal to the Stirling Numbers of the Second Kind defined by the recurrence relation

$$S(n,k) = S(n-1,k-1) + kS(n-1,k)$$

with boundary values

$$S(n,1) = 1 \ ; \quad S(n,n) = 1 \ ; \quad S(n,k) = 0 \text{ for } k > n \ .$$

Proof. All partitions of a set of n elements into k non-empty subsets can be obtained by subjoining a set which contains one element, to each partition of n-1 elements into k-1 subsets (thus getting k subsets) and by putting one element successively into each subset of each partition of n-1 elements into k subsets; i.e.,

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) \ .$$

The boundary values are obvious.

Tables of the Stirling Numbers of the Second Kind are published in [1] and [2]. A short table is given on the following page. For a closed form of $S(n,k)$ see [4].

The three kinds of equivalence relations are connected by

Theorem 4.

Let μ_1 and μ_2 be mappings of B^A. Then

$$\mu_1 \overset{t}{\sim} \mu_2 \implies \mu_1 \overset{d}{\sim} \mu_2 \implies \mu_1 \overset{p}{\sim} \mu_2 \ .$$

In other words, the three kinds of equivalence relations do not "overlap".

Proof. The theorem follows easily from Definitions 1 to 3.

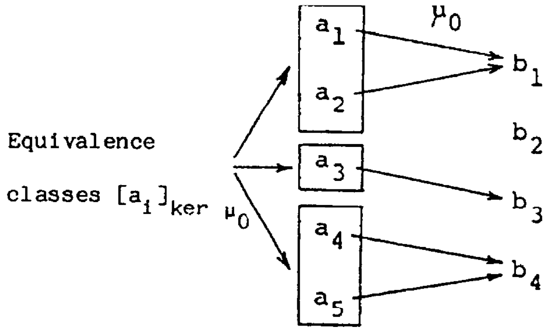
Now we know how many proto-, deutero-, and trito-equivalence classes there are in the set B^A of mappings between finite sets A and B. But how many mappings are in each equivalence class? This will be answered by the following three theorems. We begin with trito-equivalence classes in order to simplify the proofs.

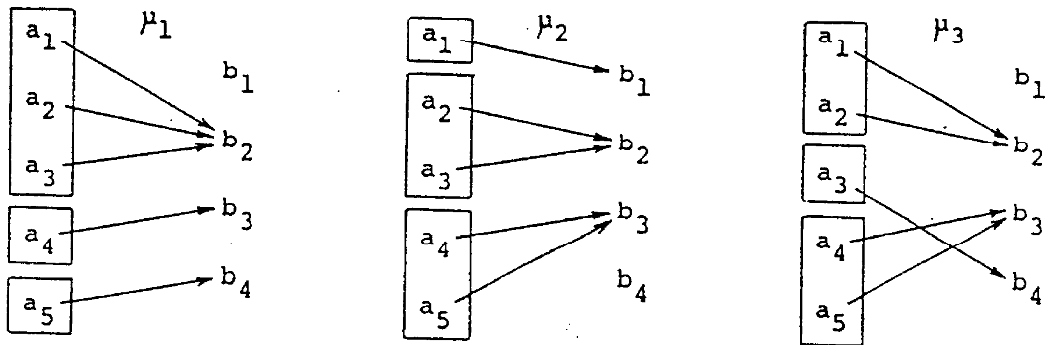
Stirling Numbers of the Second Kind $S(n,k)$

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	1	3	1							
4	1	7	6	1						
5	1	15	25	10	1					
6	1	31	90	65	15	1				
7	1	63	301	350	140	21	1			
8	1	127	966	1701	1050	266	28	1		
9	1	255	3025	7770	6951	2646	462	36	1	
10	1	511	9330	34105	42525	22827	5880	750	45	1

Example.

Let be $A = \{a_1, a_2, a_3, a_4, a_5\}$ and $B = \{b_1, b_2, b_3, b_4\}$.





$\mu_1 \overset{p}{\sim} \mu_0$ $\quad$ $\mu_2 \overset{d}{\sim} \mu_0$ $\quad$ $\mu_3 \overset{t}{\sim} \mu_0$

Notation Let be $\mu: A \longrightarrow B$. Since the cardinality of A is equal to the sum of the cardinalities of the equivalence classes in the quotient set $^A/_{\ker \mu}$, each mapping μ partitions by its kernel the cardinality of A into k parts where $k = \text{card}\ ^A/_{\ker \mu}$. These parts (the cardinalities of the equivalence classes in $^A/_{\ker \mu}$) can range between 1 and card A. Let i^{e_i} denote that there are e_i equivalence classes in $^A/_{\ker \mu}$ whose cardinalities are equal to i. Then the partition of card A = n by ker μ can be written as

$$\pi(n)_{\ker \mu} = (1^{e_1}, 2^{e_2}, \ldots, n^{e_n})$$

where $\qquad e_1 + 2e_2 + \ldots + ne_n = \text{card } A = n$

and $\qquad e_1 + e_2 + \ldots + e_n = \text{card}\ ^A/_{\ker \mu} = k$.

Example. Let n = 25 be partitioned into eight parts:

$25 = 1 + 1 + 1 + 3 + 3 + 5 + 5 + 6$

This partition can be written as

$\pi(25) = (1^3, 3^2, 5^2, 6^1)$.

Theorem 5.

Let the partition of card A = n by the kernel of a mapping $\mu : A \longrightarrow B$ be

$$\pi(n)_{\ker \mu} = (1^{e_1}, 2^{e_2}, \ldots, n^{e_n})$$

and let $[\mu]_{\underset{\sim}{t}}$ denote the trito-equivalence class of μ, i.e., $[\mu]_{\underset{\sim}{t}} \in B^A/_{\underset{\sim}{t}}$. Then

$$\text{card } [\mu]_{\underset{\sim}{t}} = \frac{m!}{(m-k)!}$$

where m = card B and $k = e_1 + e_2 + \ldots + e_n = \text{card } A/_{\ker \mu}$.

Proof. By Definition 3, all those mappings belong into $[\mu]_{\underset{\sim}{t}}$ where

a) the number k of equivalence classes in $A/_{\ker \mu}$ remains constant; this is possible for $\binom{m}{k}$ different images $\mu(A)$;

b) the elements in $\mu(A)$ are permuted; this is possible in k! ways.

Therefore,

$$\text{card } [\mu]_{\underset{\sim}{t}} = \binom{m}{k} k! = \frac{m!}{(m-k)!}$$

Theorem 6.

Let the partition of card A = n by the kernel of a mapping $\mu : A \longrightarrow B$ be

$$\pi(n)_{\ker \mu} = (1^{e_1}, 2^{e_2}, \ldots, n^{e_n})$$

and let $[\mu]_{\underset{\sim}{d}}$ denote the deutero-equivalence class of μ, i.e., $[\mu]_{\underset{\sim}{d}} \in B^A/_{\underset{\sim}{d}}$. Then

$$\text{card } [\mu]_{\underset{\sim}{d}} = \frac{n!}{(1!)^{e_1} (2!)^{e_2} \ldots (n!)^{e_n}} \binom{m}{k} \frac{k!}{e_1!\, e_2! \ldots e_n!}$$

where m = card B and $k = e_1 + e_2 + \ldots + e_n = \text{card } A/_{\ker \mu}$.

Proof. By Definition 2, all those mappings belong into $[\mu]_{\underset{\sim}{d}}$ where

a) the elements of A are permuted such that elements which belong into the same equivalence class of $A/_{\ker \mu}$ are taken as like elements; the number of these permutations is

$$\frac{n!}{(1!)^{e_1} (2!)^{e_2} \ldots (n!)^{e_n}};$$

b) the number k of equivalence classes in $A/_{\ker \mu}$ remains constant; this is possible for $\binom{m}{k}$ different images $\mu(A)$;

c) the equivalence classes in $A/_{\ker \mu}$ are permuted such that equivalence classes with the same cardinality are taken as like classes (otherwise, mappings appearing under (a) would be repeated here); the number of these permutations is

$$\frac{k!}{e_1!\ e_2!\ \dots\ e_n!}\ .$$

Therefore, the number of all mappings in $[\mu]_{\tilde{d}}$ is

$$\frac{n!}{(1!)^{e_1}\ (2!)^{e_2}\ \dots\ (n!)^{e_n}} \binom{m}{k} \frac{k!}{e_1!\ e_2!\ \dots\ e_n!}\ .$$

Theorem 7.

Let the partition of card A = n by the kernel of a mapping $\mu : A \longrightarrow B$ be

$$\pi(n)_{\ker\ \mu} = (1^{e_1}, 2^{e_2}, \dots, n^{e_n})\ .$$

Let be $k = e_1 + e_2 + \dots + e_n$ and card B = m and denote the proto-equivalence class of μ by $[\mu]_{\tilde{p}}$, i.e., $[\mu]_{\tilde{p}} \in B^A/_{\tilde{p}}$. Then

$$\text{card } [\mu]_{\tilde{d}} = \sum_{\pi(n)_{\ker\ \mu}} \text{card } [\mu]_{\tilde{d}}$$

$$= \sum_{i=1}^{P(n,k)} \frac{n\ !}{(1!)^{e_{i1}}\ (2!)^{e_{i2}}\ \dots\ (n!)^{e_n}} \binom{m}{k} \frac{k!}{e_{i1}!\ e_{i2}!\ \dots\ e_{in}!}$$

where the sum of card $[\mu]_{\tilde{d}}$ is taken over all partitions

$$\pi(n)_{\ker\ \mu} = (1^{e_{i1}}, 2^{e_{i2}}, \dots, n^{e_{in}})$$

of n into k integer parts, i.e.,

$$e_{i1} + 2e_{i2} + \dots + ne_{in} = n \quad \text{and} \quad e_{i1} + e_{i2} + \dots + e_{in} = n$$
$$\text{for } i = 1, 2, \dots, P(n,k)\ .$$

Proof. Obvious by Definition 1 and Theorem 6.

Example.

Let be $A = \{a_1, a_2, \dots a_6\}$ and $B = \{b_1, b_2, \dots b_5\}$. Then card $A = n = 6$; card $B = m = 5$; card $B^A = m^n = 15625$; $M = \min\{n,m\} = 5$; $k = 1, 2, \dots, 5$.

$\pi(n)_{\ker \mu}$	k	P(n,k)	S(n,k)	card $[\mu]_{\underset{\sim}{p}}$	card $[\mu]_{\underset{\sim}{d}}$	card $[\mu]_{\underset{\sim}{t}}$
6^1	1	1	1	5	5	5
$1^1, 5^1$ $2^1, 4^1$ 3^2	2	3	31	620	120 300 200	20
$1^2, 4^1$ $1^1, 2^1, 3^1$ 2^3	3	3	90	5400	900 3600 900	60
$1^3, 3^1$ $1^2, 2^2$	4	2	65	7800	2400 5400	120
$1^4, 2^1$	5	1	15	1800	1800	120

$$\text{card } B^A/_{\underset{\sim}{p}} = M = 5$$

$$\text{card } B^A/_{\underset{\sim}{d}} = \sum_{k=1}^{M} P(n,k) = 10$$

$$\text{card } B^A/_{\underset{\sim}{t}} = \sum_{k=1}^{M} S(n,k) = 202$$

BIBLIOGRAPHY TO APPENDIX II

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