

CHAPTER 3  
RELATING STRUCTURES TO POPULATION MEASURES

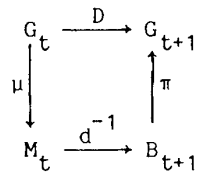
It is clear that minimal structures exist, at the very least as useful descriptive devices. Well defined structural numbers exist which represent marriage rules based on statements of kinship. This chapter discusses the relationship of structural numbers to population measures both for systems with structural numbers (i.e., kin-based marriage rules) and without structural numbers (generally, lineage organized systems).\*

Kin-Based System

A typical (proscriptive) form of kin-based marriage rule is "thou shalt not marry thy first cousin or closer relative". A person living in a culture with such a rule may think of this as an "incest prohibition". As demonstrated in the previous chapter on minimal structures, these rules have been observed to have a variety of forms, differing in different cultures.

The major form of representation of kin-based marriage rules in ethnographic descriptions is as an idealized pictorial representation of the action of a rule on a particular genealogy. By convention and for efficient use of space, ethnographers commonly adopted the pictorial representation which show the smallest population size which, if regularly following the rule, would reproduce the network of social relations of the oldest generation depicted, in the shortest possible number of generations. These minimal structures were the subject of the last chapter and of Appendix I.

An algebraic representation of the present problem is the following, adopted from Duchamp and Ballonoff (1974):



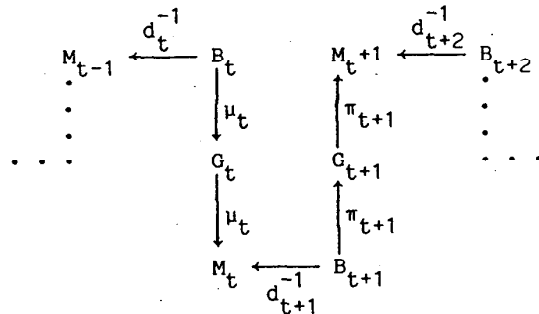

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\* Many of the statements in the main chapter are derived in more general form, or analyzed in more depth, in the chapter appendix. Note also that all of the growth rate and growth risk equations of this chapter need to be used in connection with equation (2) of Chapter 4 to generate the empirical growth rate predictions in particular applications. This was illustrated in Ballonoff (1982).

where the various sets, mappings, and inverse mappings are these:

- $G_t, G_{t+1}$  : the populations at times  $t$  and  $t+1$  respectively
- $D$  : evolutionary operator describing changes in population biology
- $B_{t+1}$  : a partition of  $G_{t+1}$  into sets of persons with the same parents
- $M_t$  : the sets of mated couples at time  $t$
- $d$  : an injection, corresponding to ancestor
- $d^{-1}$  : a surjection, corresponding to descent
- $\pi$  : a partition of  $G$
- $\mu$  : a surjection
- $\mu^{-1}$  : an injection

Population theory as usually formulated is interested in properties of  $D$  and of changes in the properties of  $G$  under the action of  $D$ . On the other hand, social theory is commonly concerned with all the other mappings, particularly  $d$ . In fact, social theory, and particularly ethnography and ethnology, frequently know nothing at all about  $D$ , so that a more accurate picture of the state of ethnographic knowledge is something like



Knowledge of the maps  $\mu$  and  $\pi$  are essentially that not everyone gets married ( $\mu$  is injective) and that everyone is the offspring of someone ( $\pi$  is surjective). A useful property of  $d^{-1}$  is that essentially  $d^{-1}$  assigns sets of offspring to sets of parents. Since every set of offspring has parents, but some sets of parents may be non-reproductive,  $d$  is injective and  $d^{-1}$  is surjective. Furthermore,  $d_t^{-1}\pi_t$  is the partition of  $G_t$  induced by a particular choice of parentage.

It is thus possible to ask how many different ways parentage may be assigned. Two assignments are indistinguishable if and only if: each assignment results in the same set of parents being assigned offspring; and, the equivalence classes induced on the offspring are the same. That is, two mappings are equivalent for our purposes if they always pick the same set of parents, and the same groupings of offspring, even if some mappings reassort which groupings of offspring are assigned to which set of parents. Therefore, one gets demographic information, but no genetic information transmitted through the maps  $d$ .

A theorem has been proven (See Schadach, 1967 and Appendix II) showing that the total number of different (non-equivalent as defined above) mappings satisfying this criteria is given by

$$S(g,k) \binom{m}{k} \tag{1}$$

for each value  $1 \leq k \leq k_{\min}$ , where  $g$  = cardinality of  $G_t$ ,  $m$  = cardinality of  $M_{t-1}$ , and  $k_{\min} = \text{minimum}(g,m)$ . Note that  $g = N_t$  is the population size, and  $S(g,k)$  is the Stirling Number of the Second Kind (See also the appendix to Ballonoff, 1983 for other references and related discussion. The critical source is Schadach's Corollary III to his Theorem II of Appendix II below).

Therefore, USE OF STIRLING NUMBERS OF THE SECOND KIND ARE REQUIRED BY THE EXISTENCE OF MARRIAGE RULES. More precisely, minimal structures and the algebraic representation of them, imply this statistic is necessary. The Stirling Number result is not an assumption of the theory, but a consequence of ethnographic description. Without knowledge of this particular proof, Ballonoff (1973 and later) used the Stirling Number of the Second Kind to compute theoretical estimates of the population statistics of the Western Apache, the U.S. 1970 census, the Hopi, and other populations. These computations were made possible by use of the characteristic properties of the Stirling Number of the Second Kind which I discuss next.

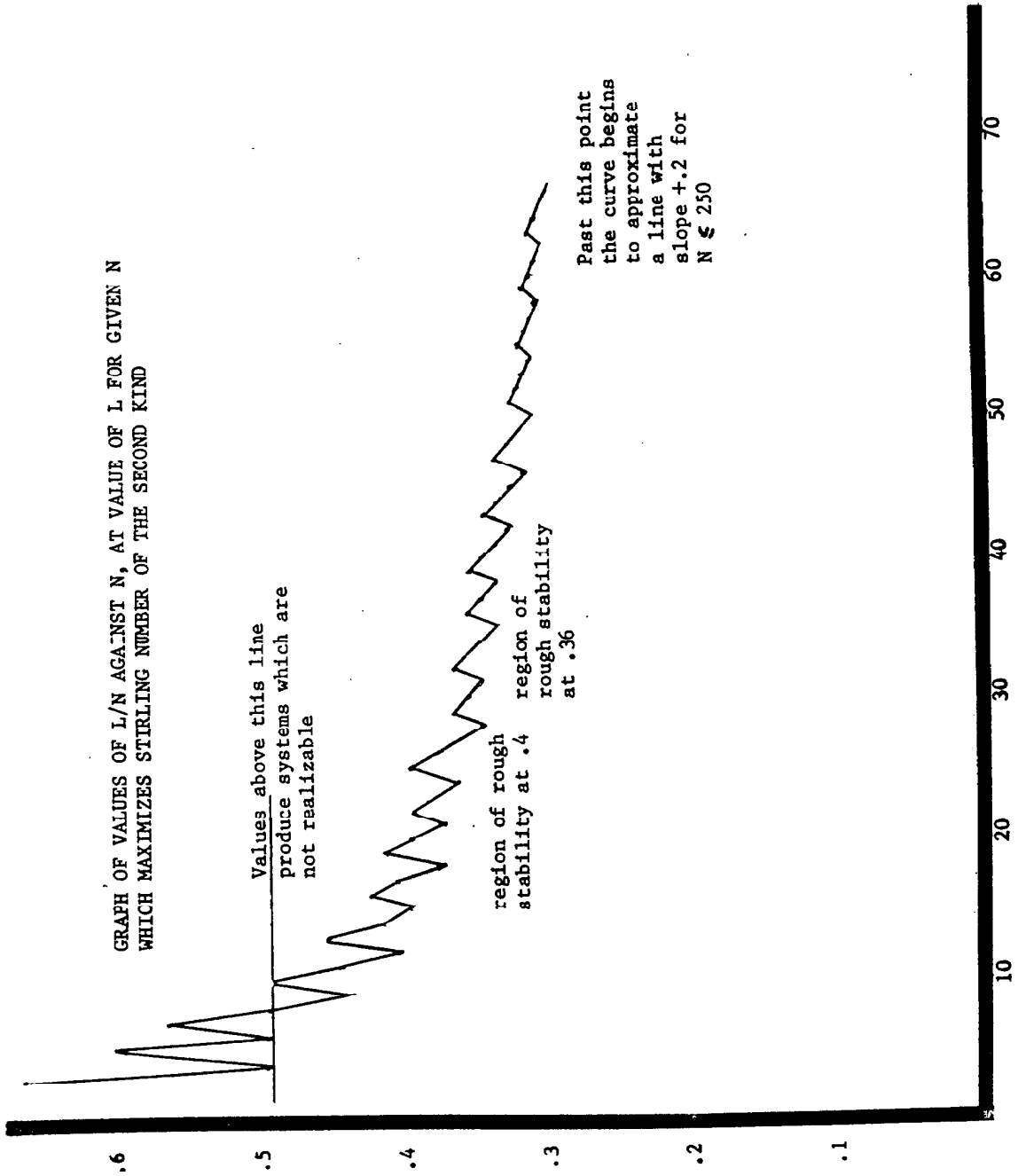
Note from published tables (such as Andrew, 1965) that almost the entire distribution of values of the Stirling Number of the Second Kind are concentrated at or very close to a particular value of  $k$  for given value of  $g$  of equation (1). This fact allows one to predict, for each possible population size, a particular number of sets of parents which would be associated with producing a population of the given size. In my papers these values are denoted  $N$  = population size, and  $L$  = the value of  $k$  for which the Stirling Number is maximized for fixed value of  $N$ . Based on the tables of Andrew, 1965, tables of the pairs of values  $(N,L)$  were published for  $N \leq 100$  in Ballonoff (1975) and for  $N \leq 248$  in Ballonoff (1976).

These tables reveal properties of  $L$  and  $N$ . The ratio of  $L/N$  is highly unstable, but for  $N > 25$  is clearly decreasing at a relatively constant rate (see graph below). The inverse ratio of that on the graph is  $N/L$ , which may be interpreted as the average family size. Note that this value increases with  $N$ . The proportion of reproductive females is proportional to, indeed estimated by,  $1/2(L/N)$ . In fact in the present case

$$p = \frac{2L}{N}$$

$$n = \frac{N}{L} \tag{2}$$

GRAPH OF VALUES OF L/N AGAINST N, AT VALUE OF L FOR GIVEN N  
WHICH MAXIMIZES STIRLING NUMBER OF THE SECOND KIND



where  $n$  = average family size and  $p$  = proportion of adult population ascribed as reproducers, and  $np = 2$ . For  $N < 10$  there are values of the ratio which would force  $p > 1.0$ , which would be physically impossible. For  $N > 25$ , there is a quite evident regular decline in the proportion. Only for  $15 \leq N \leq 25$  is the ratio apparently stable in the region, and also without impossible values. At much larger values of  $N$ , the proportion  $p$  eventually falls toward zero.

The characteristics of the Stirling Number can be used to derive properties of populations (of any size) following rules which have minimal representations of particular sizes, much as chemical theory can know the properties of large bulks of materials by knowledge of the properties and sizes of individual atoms or molecules. This is summarized in the following paragraphs. Note, however, that for populations to be stable in size, growth rates must be zero, and for social systems to be "stable", their network characteristics must be relatively constant.\* Furthermore, in observed human systems, stable family sizes (predicted here as  $n$ ) do not tend toward infinity, nor does the proportion of married females (predicted here as  $p$ ) tend toward zero in large populations. Also, systems must avoid the possibility of requiring  $p > 1$ .

The structural number of a rule is the number of marriages in the minimal representation, per generation. This can be written, for a minimal system,

$$s = \text{cardinality of } M \tag{3}$$

(We can suppress the time subscript since in the minimal representations of rules all generations have the same number of marriages). Another interpretation of  $s$  is therefore the following: if in some imaginary system,  $M$  is the number of families, and  $M_0$  the number of families that do not reproduce females who in turn reproduce, then

$$s \leq M - M_0 \tag{4}$$

Using a binomial model for the distribution of sexes into families, and looking only at the minimum bound of (4), Ballonoff (in an article in Ballonoff, 1974a) used a binomial estimate of  $M_0$  to obtain

$$s = M - M \left(1 - \frac{1}{M}\right)^F \tag{5}$$

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\* I have outlined an approach to the network analysis of structures subject to marriage rules, i.e., minimal structure restrictions, in Ballonoff 1976b.

Equation 5 can be solved for a particular value of  $M$  given a particular  $s$ , where the second term on the right in (5) is an estimate of the value of  $M_0$  and  $F$  is the number of reproductively active females in a minimal system. (Another derivation of this form and interpretation of its meaning is given in Chapter 4).

One condition for solving (5) can be that  $F = M$ . Solving (5) with this as a side condition produces for each  $s$  one value of  $M$ . This is the value listed in the table below.

TABLE 1  
TABLE OF STATISTICS REFERENCED IN TEXT  
FOR STRUCTURAL NUMBERS  
 $s \leq 10$

<u>s</u>	<u>n</u>	<u>p</u>	<u>N</u>	<u>M or L</u>
2	2.0	1.0	4	2
3	2.0	1.0	8	4
4	2.18	.92	13	6
5	2.43	.82	17	7
6	2.44	.82	22	9
7	2.70	.74	27	10
8	2.75	.73	33	12
9	2.92	.68	38	13
10	2.92	.68	44	15

We can now note that the interpretation  $M = f$  just given, is the same as setting  $M$  equal to the number of one-female reproductive units. The interpretation of  $M$  was purely in terms of the behavior of populations following a rule with structural number  $s$ , and in fact is independent of population size except when actual size approaches  $s$ . Interpretation of  $L$  is in terms of its relationship to a value  $N$  which together are used to compute two numbers  $n$  and  $p$  satisfying  $np = 2$  subject to the particular rule. Therefore: The  $n$  and  $p$  values thus determined are independent of empirical population size except in the case where the empirical size is of the order of either the size  $N$  or of  $2s$ . However,  $n$  and  $p$  do depend on the marriage rule which has structural number  $s$ , used to find the  $n, p$  pair characteristic of that  $s$ . This result holds for structural numbered systems with no lineage organization. For lineage systems, however, as discussed in the appendix to Ballonoff, 1982, the actual population size relates to the structural description.

We can note some properties of the table above. First, all values listed, i.e., all solutions to (5) with the side condition  $F = M$ , produce values of  $p$  which are feasible. None of these solutions produce values which are physically impossible by previous interpretations. Further, the values of  $4 \leq s \leq 6$  fall in the stable range of the curve of  $L/N$ .

The importance of this fact is most empirically known marriage rules have structural numbers in this range. A marriage rule such as "thou shalt not marry that first

cousin or closer relative" has structural number  $s = 4$ , forced marriage to a double second cousin also has  $s = 4$ , third cousin forced rules can require as little as  $s = 5$ , and even the most complicated clan systems studied by White (1963) have groups of the order  $s = 6$ . All of these rules however, can and do occur in real populations far larger than their minimal sizes.

Most known systems, and especially most large systems in fact have rules  $s = 4$  or so. As an exercise in how to use structural numbers to compute statistics for large systems, consider that approximately 2/3 of the states of the U.S. prohibit first cousin and closer marriage ( $s = 4$ ) while the remaining 1/3 prohibit only closest matings (brother-sister or parent-offspring, with  $s = 2$ ). This means that taking a weighted average of about 2/3 of the values for  $n$  and  $p$  listed under  $s = 4$  in the table, and about 1/3 of the values for  $s = 2$ , produce an estimate of the average completed family size and proportion of females ever married for the 1970 U.S. census, widely acclaimed as a time when the U.S. was at zero demographic growth. The fact that the U.S. can be at equilibrium with family sizes close to 2.1 is a result of the fact that the structural numbers characterizing American marriage are small, and are in particular (approximately) 2/3 of the time ( $s = 4$ ) and 1/3 of the time ( $s = 2$ ). More discussion of this is found in Ballonoff (1976), pages 94, and 98-99 for the U.S.; and for the Apache, in Ballonoff (1973).

I am therefore able to make an important claim: that "ordinary" demographic methods for estimating equilibrium family size only work by accident. Normal population methods are dependent on stationarity properties of probability models using age structured birth and death rates. Social structure as studied here has no formal part in demographic theory. The reverse is also true, since the present theory greatly abstracts all age structures into "generations". Therefore, one of the most powerful conclusions of this theory, elaborated in the appendix to this chapter, in Chapter 4, and elsewhere (particularly Ballonoff 1982, 1982a) is that population growth has predictable, measurable, causes in cultural structures, which causes are not known, measurable nor predictable from normal demographic models.

#### Lineage Systems

To understand how the distribution of the Stirling Number of the Second Kind can be used as a predictive tool for lineage systems, an example is useful. Consider the distribution of the number of combinations of  $N = 50$  unique objects into  $1 \leq L \leq N$  indistinguishable cells such that no cell is empty. At each end, the absolute frequency is  $10^0 = 1$  (for  $L = 1$  and also  $L = N$ ). At the peak the absolute frequency is  $10^{46}$  ways. If this statistic is taken as a density function, the prediction of modal frequency of  $L$  is the maximum value of the absolute frequency distribution (here, at  $L = 16$ ). In this case, any number near the maximum  $L \pm 2$  is almost equally likely; but from here, a drop in expected frequency is sharp, by a factor of 100, out to about  $\pm 6$  from the peak (the range from  $L = 10$  to  $L = 23$ ). Beyond these

values, the drop is so steep in absolute frequency, that in relative frequency terms, there is almost no probability of their existence. Similarly, there is a small and rapidly decreasing probability from the  $\pm 2$  range to the  $\pm 6$  range.

The interpretation of the relation  $np = 2$  is now relatively simple. For each  $N$  the unique value of the maximum frequency is taken as the predictor of a unique  $L$ . Then the average number of offspring  $n$  per family unit is  $n = N/L$ . The average number of females (or, males) in a bi-sexual population can be estimated by  $\frac{1}{2}N$ , so that the ratio of reproducers to offspring of the same sex is  $L/\frac{1}{2}N = 2L/N = p$ . Thus clearly  $np = 2$ . This is a result of the interpretation of the Stirling Number of the Second Kind, not a derivation from some concept other than the interpretations given.

The attached table 2 shows the values of  $L$  thus found for values of  $N$  from 1 to 100, together with the corresponding values of  $n$  and of  $p$ . Also note that for each  $n$  there is a unique  $L$ , but for each  $L$ , several  $N$  (in sequence) may be associated. Thus, prediction up, from  $N$  to  $L$  values, gives unique values, but prediction down, from  $L$  to  $N$  values, produces a range or "fan" of values.

There are therefore two distinct sources of growth risk that can result from the fact that the Stirling Number of the Second Kind underlies human lineage systems, and therefore is fundamental to much of human cultural diversity. One of these is the risk that can result from the "fan effect". If a system were "trying to maintain stability of  $L$ " (i.e., if the number of named lineages), for most values of  $L$  a range of values of per generation population size  $N$  (and hence of  $p$  and of  $n$ ) are compatible with the given  $L$ . The existence of variation in the possible pairs of  $n$  and  $p$  values results in growth risk (or, decline) in population size as a result of seeking stability of  $L$ . (This was studied in the appendix to Ballonoff, 1982).

The second form of risk results from the properties of the density function given by the Stirling Number near its peak. For at least some  $N$  the probability (i.e., possibility density) of values close to the peak  $L$  are nearly identical with the probability density of the peak  $L$ . Therefore, preservation of  $L$  by maintenance of  $N$  (i.e., maintenance of the population size) does not of itself guarantee that the right number of  $L$  will be reproduced.\* However, it clearly may be one reason that various cultural devices exist for identifying "kin" in groups other than the immediate local population or tribe. Further, the combination of these two forms of

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\* Because of the analytical perversity of the expression for the Stirling Number of the Second Kind, I have not studied this topic in depth, mathematically. This is also why I have not attempted a distributional or "variance" analysis of this statistic. While it is extremely perverse mathematically, examination of tables of the statistic easily lead to the conclusion that, as a density function, it is highly concentrated; hence the variance of the prediction is quite low.



TABLE 2: TABLE OF THE VALUE OF L WHICH OCCURS AT THE MAXIMUM DENSITY OF NUMBER OF COMBINATIONS, FOR  $1 \leq N \leq 100$ .  
 Columns are: (1) N, (2) L at maximum density, (3)  $p = 2L/N$ , (4)  $n = N/L$ .

(1)	(2)	(3)	(4)	(1)	(2)	(3)	(4)
1	1	2.0000	1.0000	51	17	0.6667	3.0000
2	2	2.0000	1.0000	52	17	0.6538	3.0588
3	2	1.3333	1.5000	53	17	0.6415	3.1176
4	2	1.0000	2.0000	54	17	0.6296	3.1765
5	3	1.2000	1.6667	55	18	0.6545	3.0556
6	3	1.0000	2.0000	56	18	0.6429	3.1111
7	4	1.1429	1.7500	57	18	0.6316	3.1667
8	4	1.0000	2.0000	58	18	0.6207	3.2222
9	4	0.8889	2.2500	59	19	0.6441	3.1053
10	5	1.0000	2.0000	60	19	0.6333	3.1579
11	5	0.9091	2.2000	61	19	0.6230	3.2105
12	5	0.8333	2.4000	62	19	0.6129	3.2632
13	6	0.9231	2.1667	63	20	0.6349	3.1500
14	6	0.8571	2.3333	64	20	0.6250	3.2000
15	6	0.8000	2.5000	65	20	0.6154	3.2500
16	7	0.8750	2.2857	66	20	0.6061	3.3000
17	7	0.8235	2.4286	67	21	0.6269	3.1905
18	7	0.7778	2.5714	68	21	0.6176	3.2381
19	8	0.8421	2.3750	69	21	0.6087	3.2857
20	8	0.8000	2.5000	70	21	0.6000	3.3333
21	8	0.7619	2.6250	71	22	0.6197	3.2273
22	9	0.8182	2.4444	72	22	0.6111	3.2727
23	9	0.7826	2.5556	73	22	0.6027	3.3182
24	9	0.7500	2.6667	74	22	0.6046	3.3636
25	10	0.8000	2.5000	75	23	0.6133	3.2609
26	10	0.7692	2.6000	76	23	0.6053	3.3043
27	10	0.7402	2.7000	77	23	0.5974	3.3478
28	10	0.7143	2.8000	78	23	0.5897	3.3913
29	11	0.7586	2.6364	79	24	0.6076	3.2917
30	11	0.7333	2.7273	80	24	0.6000	3.3333
31	11	0.7097	2.8182	81	24	0.5926	3.3750
32	12	0.7500	2.6667	82	24	0.5854	3.4167
33	12	0.7273	2.7500	83	25	0.6024	3.3200
34	12	0.7059	2.8333	84	25	0.5952	3.3600
35	12	0.6857	2.9167	85	25	0.5882	3.4000
36	13	0.7222	2.7692	86	25	0.5814	3.4400
37	13	0.7027	2.8462	87	25	0.5747	3.4800
38	13	0.6842	2.9231	88	26	0.5909	3.3846
39	14	0.7179	2.7857	89	26	0.5843	3.4232
40	14	0.7000	2.8571	90	26	0.5778	3.4615
41	14	0.6829	2.9286	91	26	0.5714	3.5000
42	14	0.6667	3.0000	92	27	0.5870	3.4074
43	15	0.6977	2.8667	93	27	0.5806	3.4444
44	15	0.6818	2.9333	94	27	0.5745	3.4815
45	15	0.6667	3.0000	95	27	0.5684	3.5185
46	15	0.6522	3.0667	96	28	0.5833	3.4286
47	16	0.6809	2.9375	97	28	0.4773	3.4643
48	16	0.6667	3.0000	98	28	0.5714	3.5000
49	16	0.6531	3.0625	99	28	0.5657	3.5357
50	16	0.6400	3.1250	100	28	0.5800	3.4483

growth risk go a long way toward understanding much of lineage organization, expansion, times of split into subgroups, and numerous other questions of social dynamics (see Ballonoff, 1983).

Notice carefully that neither of the above growth risks are the growth rates computed from the equation developed later for changes in or mixes of rules; though both are founded in properties of the Stirling Number. Thus, mixture of cultural forms, and cultural change, is a distinct form of growth risk from that introduced by essentially stochastic properties of attempting stability of certain L or N values. Further, none of the culturally determined growth risks (i.e., arrived at by analysis of the Stirling Number results), is the risk or growth or decline resulting from sex-ratio disturbances, nor of age-structure dependent distributions.

One of the most important results of this theory is therefore its ability to identify and quantitatively study, causes of population growth not previously identified by demographic theory; and which result directly from analysis of the cultural system. (See Ballonoff, 1982, 1982b, and 1983 for detailed developments).

#### Marriage Rules and Growth

In discussions above, one result of interpreting the Stirling Number of the Second Kind was development of an equation of the form " $np = 2$ ". This can obviously also be written as  $\frac{1}{2}np = 1$ . Since, with exponential term  $r$  equal to zero,  $e^r = 1$ , a useful form to look at is equation (7) below.

$$\exp(rT) - \frac{1}{2} np \tag{7}$$

In (7)  $n$  and  $p$  are computed instantaneously as properties of the marriage rule of the population of reproducing age,  $r$  is a growth or decline rate per year, and  $T$  is the generation interval in years. Note that if  $T$  is taken as a unit generation interval, then  $rT$  is the per generation growth rate.

I here note implications of equation (7) for population growth, given that  $n$  and  $p$  depend on the marriage rule as determined by ethnographer description. Thus, as explored in Ballonoff (1982) a shift in social structure, in particular in marriage rules, will result in population growth when the change is not instantaneous and complete, and the amount of growth that results is (essentially) independent of the values of  $n$  and  $p$ , but is highly dependent on the pattern shift and its speed relative to the generation interval in years. (This shift pattern itself may also be dependent on the particular rules being adopted or abandoned.) The resulting growth is computed in units of  $T$ , i.e., of the generation interval.

Let  $v_a(t)$  be the relative proportion of a particular cultural group following rule  $a$  at time  $t$ , let  $v_b(t)$  be the relative proportion following rule  $b$ , and let  $v_a(t) = 1 - v_b(t)$ , which thus assumes that there are only two rules to consider and that the population is following one or both in some proportion. (A more general form is found in Ballonoff, 1982). I shall use subscripted variables  $n_a$ ,  $p_a$ , etc. to show the values of  $n$  and  $p$  computed for rules  $a$  and  $b$ , etc. Note that  $n_i p_i = 2$  for any

given rule  $i$ , but that in general  $n_i p_j \neq 2$  if  $i$  and  $j$  do not denote the same or isomorphic rules.

Assume that mixed sets of rules being followed by single cultural units allows the average value of the entire unit to be computed as linear combination of the average values of its component rules. Then I interpret  $v_a(t)$  and  $v_b(t)$  as the degree to which the population is following rules  $a$  and  $b$  at time  $t$ , and compute the expected values  $n(t)$ , and  $p(t)$ , of  $n$  and  $p$  for the cultural unit at time  $t$  as follows:

$$n(t) = v_a(t)n_a + v_b(t)n_b = v_a(t)(n_a - n_b) + n_b \quad (8)$$

$$p(t) = v_a(t)p_a + v_b(t)p_b = v_a(t)(p_a - p_b) + p_b \quad (9)$$

Rewriting equation (7) using  $n(t)$  and  $p(t)$  gives at equilibrium

$$\exp(rT) = \frac{1}{2}n(t)p(t) \quad (10)$$

which transforms, using equations (8) and (9) and the identities  $n_i p_i = 2$ , into

$$\begin{aligned} \exp(rT) = 1 - \frac{1}{2}v_a(t)^2 n_b p_a + \frac{1}{2}v_a(t)n_b p_a - 2v_a(t) \\ - \frac{1}{2}v_a(t)^2 n_a p_b + \frac{1}{2}n_a(t)n_a p_b + v_a(t)^2. \end{aligned} \quad (11)$$

Notice that equation (11) implies that  $r = 0$  (for finite non-zero  $T$ ) if and only if either  $v_a(t) = 0$  or  $v_a(t) = 1$ , i.e., that the growth rate will be zero only if the population is following only one rule, or at least only rules with identical structural numbers.

In order to study shifts and patterns of shifts between rules, I adopt the convention that rule  $a$  is the initial rule and  $b$  the final rule. Because of the above mentioned property of equation (11), whenever a population is shifting between rules, or using a mixture of rules for any reason, there will be growth, and this growth occurs in spite of the fact (or because of it) that the population is attempting to act at equilibrium with respect to each rule it follows, to the degree to which it follows that rule.

For example, if  $p_a = 1$ ,  $n_a = 2$ ,  $p_b = .5$ ,  $n_b = 4$  and  $v_a(t) = .5$  then at time  $t$  the growth rate per generation  $rT = .117$ , or with a generation interval  $T = 15$  years,  $r = .023$  growth per year in year  $t$ . Within ranges of  $n$  and of  $p$  likely to occur in human populations, it appears that values of  $rT$  computed from equation (11) are very sensitive to the particular choices of initial and final values, as is demonstrated in tables 3.1 a, b, and c below. In each case the initial values selected correspond to a rule which only prohibits marriage between brothers and sisters. Table 3.1a corresponds to shift to a final rule which prohibits all first cousin marriages in addition to all brothers and sisters. Table 3.1b corresponds approximately to a

shift to a rule which would prohibit all marriages as close or closer than second cousins. Table 3.1c would thus reflect a change to a very complex rule (such as extended lineage exogamy with numerous lineages).

Table 3.2 illustrates the compounding effects of the shift described in table 3.1c, from a simple to a complex system. The table assumes that this change occurs at a slow pace bordering on social stagnation (if  $T = 25$  years, ten generations is 250 years). Yet in that situation, the total growth which results is nearly 100 times greater than if the change occurs in one generation, resulting in a more than doubling of the population. When change is not at a uniform rate in  $v_a(t)$ , as above, the precise pattern of shift will determine the resulting cumulative growth.

A major result therefore is that the longer the total real elapsed time measured in number of generations for a shift to occur, the greater will be its cumulative effect on population size before values appropriate for the new cultural system are reached. In general, the growth rate will be a maximum whenever any two rules in use have  $v_i(t) \approx .5$ . This fact implies that one way to minimize growth while shifting from one rule to another may be for a system to use more than two rules at particular times in the period of shift.

TABLE 3.1

EFFECTS OF SHIFTS IN VALUES OF  $n$  AND OF  $p$  UNDER TWO DIFFERENT ASSUMPTIONS OF FINAL MARRIAGE RULES, STARTING FROM THE SAME INITIAL MARRIAGE RULES, USING EQUATION (11) AND NUMERICAL VALUES STATED IN THE TEXT. THE NOTATIONS  $n(t)$ ,  $p(t)$  MEANS THE AVERAGE VALUES OF  $n$  AND  $p$  AT INDICATED  $v(t)$  VALUES.

TABLE 3.1a

$$p_b = .92$$

$v_a(t)$	$r(t)T$	$n(t)$	$p(t)$
.1	.00063	2.017	.992
.2	.00111	2.035	.984
.3	.00146	2.052	.976
.4	.00166	2.070	.968
.5	.00173	2.087	.960
.6	.00166	2.104	.952
.7	.00146	2.122	.944
.8	.00111	2.139	.936
.9	.00063	2.157	.928
1.0	.0	2.174	.920

TABLE 3.1b

$p_b = .82$

$v_a(t)$	$r(t)T$	$n(t)$	$p(t)$
.1	.00354	2.044	.982
.2	.00630	2.088	.964
.3	.00826	2.132	.946
.4	.00944	2.176	.928
.5	.00983	2.220	.910
.6	.00944	2.263	.892
.7	.00826	2.307	.874
.8	.00630	2.351	.856
.9	.00354	2.395	.838
1.0	.0	.2439	.820

TABLE 3.1c

$p_b = .50$

$v_b(t)$	$r(t)T$	$n(t)$	$p(t)$
.0	.0	4.0	.50
.1	.0440	3.8	.55
.2	.0770	3.6	.60
.3	.0988	3.4	.65
.4	.1133	3.2	.70
.5	.1177	3.0	.75
.6	.1133	2.8	.80
.7	.0988	2.6	.85
.8	.0770	2.4	.90
.9	.0440	2.2	.95
1.0	.0	2.0	1.0

TABLE 3.2

EXAMPLE OF A PARTICULAR HISTORY OF GROWTH OCCURRING  
AT AN EVEN RATE OVER TEN GENERATIONS

$r(t)T$	Time in Units of T	New Growth	Cumulative Growth
.044	1.0	.044	1.044
.077	1.0	.080	1.124
.100	1.0	.112	1.236
.113	1.0	.140	1.376
.118	1.0	.162	1.538
.113	1.0	.174	1.712
.100	1.0	.171	1.884
.077	1.0	.145	2.027
.044	1.0	.089	2.118

Also note that the greatest cumulative population growth may well occur long after the greatest amount of social change which was its cause has already passed (see Table 3.2). Measured as a percentage of the original population, the greatest amount of growth will only appear after the rate of growth in units of  $rT$  has already begun to decline. Histories of slow initial change followed by rapid completion should show rapid or relatively rapid population growth in the period of slow change, followed by little or no apparent growth in the period of rapid social change. Thus, attempts to "explain" changes in population size or in growth rate per year  $r$  as time dependent linear or log-linear functions, which are the most common models, are bound to failure: given a particular set of rules the underlying rates in terms of the  $v(t)$  are then fixed, but the particular time path of the  $v(t)$  will completely determine the ultimate growth effect.

The above equations can be derived in a more general form. If a cultural system is either in the process of transformation from one cultural pattern of marriage and kinship to another, or for some other reason using a mixed set of rules of marriage rather than a single rule, then we can denote the degree to which a population follows a rule with structural number  $i$  in time  $t$  as  $v_i(t)$ , where  $0 \leq v_i(t) \leq 1$  and

$$\sum v_i(t) = 1$$

$$n(t) = \sum v_i(t)n_i \tag{12}$$

$$p(t) = \sum v_i(t)p_i$$

where summations are taken over the set  $\{k\}$  of the  $k$  rules relevant to this particular system.

Using equations (10) and (12) we get

$$\exp(r(t)T) = \sum_{i=j} v_i^2 + \frac{1}{2} \sum_{i \neq j} v_i(t)v_j(t)n_i p_j \tag{13}$$

Equation (13) expresses the growth rate of the population as a function of what its population statistics would be if it were acting at equilibrium with respect to each rule it is attempting to follow, to the degree to which it is following that rule.

Equation (13) has the property that population growth must occur whenever more than one rule (more precisely, rules with more than one structural number among them) is being used. Since the  $n_i$  and  $p_i$  values are parameters which depend on the rules being used but not on time, the problem of minimizing population growth  $r(t)$  is the problem of finding a vector  $\underline{v}(t)$  which minimizes  $r(t)T$  at time  $t$ , or to find a sequence of such vectors which minimizes total growth over some given time period.

This appears to be a classical problem in optimization, but that is not quite true. Equation (13) or its variants may be restated as a quadratic form

$$\exp(r(t)T) = \underline{v}(t) \underline{C} \underline{v}(t)' \quad (14)$$

where  $\underline{C} = [\frac{1}{2}c_{ij}]$  is the matrix of coefficients  $c_{ij} = n_i p_j$ . Since  $n_j = 2/p_j$ , then  $c_{ij} = 2p_i/p_j$  and  $c_{ji} = 1/c_{ij}$ . Therefore  $\underline{C}$  is a linearly dependent system (determinant of  $\underline{C}$  is zero) and most of the "easy" techniques of optimization theory are not available. In fact the largest rank matrix  $\underline{C}$  which is non-singular has rank  $k = 1$ , i.e., the case of one rule and zero growth.

We could convert  $\underline{C}$  to a symmetric matrix  $\underline{P}$  by noting that in the expansion of equation (13) the term  $v_i(t)v_j(t)$  has the coefficient  $\frac{1}{2}c_{ij} + \frac{1}{2}c_{ji} = (p_i^2 + p_j^2)/p_i p_j$ . Therefore let the  $i,j$  entry of  $\underline{P}$  be  $\frac{1}{2}(p_i^2 + p_j^2)/p_i p_j$ . Note that the diagonal entries of both  $\underline{P}$  and  $\underline{C}$  are  $p_{ii} = c_{ii} = 1$ . The new matrix  $\underline{P}$  has determinant equal to zero if  $p_i = p_j$  for all  $i$  and  $j$ , but otherwise may or may not be linearly dependent, according to the values of the various  $p_i$ , i.e., according to the specific set of rules in use.

In other words, the specific social structures associated with the sets of rules in use determine the form of the mathematical solution to the problem of determining the minimum growth path.

This same conclusion may be drawn by treating the problem from another viewpoint. Let

$$d(v_i(t))/dt = \sum_j a_{ij}(v_j(t)) \quad (15)$$

or, in words, I assume first order linearity in the rate of change of the degree of use coefficients  $v_i(t)$ . Because of (4), equation (15) implies that

$$\sum_j a_{ij}(v_i(t)) = 0 \quad (16)$$

This superficially leads to use of the Chapman-Kolmogorov equations with the term in the mean and the total differential both equal to zero. (See Ballonoff (1982) for discussion of the meaning of the eigenvalues which result from this interpretation). Since precisely such models are well developed (for example in population genetics to study evolution of gene frequencies), the mathematical problem would seem to be solved, but for one flaw: the meaning of " $a_{ij}$ " in the context of minimal structure theory.

Several interpretations are available. I write " $a_{ij}(t)$ " to discuss these.

Case A: the  $a_{ij}(t)$  are completely subjective, or subject to manipulation by administrative fiat, psychological controls, or other devices. In this case we can apply classical control theory with the  $a_{ij}(t)$  considered as the policy instruments. This may be mathematically nice, but may be culturally brutal--people's marriage choices are not so easily manipulated;

Case B: the  $a_{ij}(t)$  are independent of  $t$ . This is essentially a Markovian or Chapman-Kolmogorov approach, and embodies a significant problem: in policy formulations, we are generally seeking minimum growth paths, and preferable zero growth solutions, whereas this approach leads to study of steady-state distributions of the vectors  $\underline{y}(t)$  which may well be associated with high growth rates, or with undesirable time paths of growth.

Case C: the  $a_{ij}(t)$  are really functions of  $i, j, t$  and  $T$ ; i.e., the path of change is dependent on the particular rules, the generation interval, and the time (in units of the generation interval) since change was initiated. This case is anthropologically the most realistic, since it says that the specific social networks that may be accessible at a point in time depend on the specific historic situation regarding the rules of network formation. The work of Ballonoff (1976), chapters 4 and 6 strongly suggests that this is the correct interpretation. It is, however, neither the Markovian nor Kolmogorov nor control theory formulation.

Thus the present theory may lead to a mathematically tractable tool for population planning, but the development of this tool is not simply a matter of applying techniques known from established disciplines. As with other sciences, the specific mathematical demands of an adequate theory reflect the nature of the underlying phenomena which generate them. In the case of social anthropology, the complexity of real world relationships is reflected in the linear dependence of equations (14) and its variants, and in the fact that the "simple" mathematical "solutions" (of stationarity of distributions) offered by for example the Kolmogorov backward equation would be accompanied in the real world with a nasty "side effect": population growth.

Besides the normally recognized "demographic" growth (or stationarity of population distributions) associated with study of age dependent fertility and mortality (which are not even part of the theory of minimal structures), there are at least three other causes of population growth: (1) the growth predicted by equations such as discussed above -- this may be considered "morphological" growth in that it derives from the logical or rule-like structure of the culture; (2) that which results from the "stochastic" or distributional properties of the underlying statistic, as discussed in the appendix to Ballonoff (1982) and briefly above, and which also relates to the "risk taking ability" of the culture (on lines discussed for example by Posner (1980)); (3) that which results from (essentially) sex ratio flux in finite populations. Of these three, the first and second are unknown outside of the



present theory, while the third is not a simple extension of probability theory (Ballonoff, 1976, chapter 5 treats this briefly).

Quite clearly, our concepts of how to scientifically analyze cultural systems are in need of expansion.