CHAPTER 6

TIME, FORM AND NUMBER

Introduction

The substantive development of this text was completed with the last chapter. This chapter discusses some more subtle details of the formulation, particularly of Chapter 3. Unlike earlier chapters, it does not lay out essential new theory. This chapter attempts to make the reasons for my choice of how to construct the theory more apparent.

The idea of "time" enters social-mathematical works from their earliest formulations. This is in no small part because living things are born at one time, die at another, and therefore the set of specific beings of a given system which simultaneously exist, changes. This fact has the consequence that we may construct specific histories for any simultaneously existing set of individuals. "Time" necessarily enters at this very early stage, forcing description of subject matter in an historic sense, forcing us to determine simultaneity even where other historic questions are absent.

Thus in selecting a notion of population, for whatever reason, we are simultaneously selecting some conception of time. The reasons for this are two: simultaneity of existence, and succession. In the simplest population models (discrete generation models) a particular set of individuals is conceived as being born simultaneously, dying simultaneously, and performing simultaneously all acts of interest between birth and death. At the death of one generation, another simultaneously replaces it by descent. In such a model, we may say whether or not individuals belong to the same generation. The relationship "descent" between whole generations (or between individuals as members of generations) in connection with the relation of simultaneity of existence, says of any two individuals whether or not one occurs "earlier" or "later" in some particular sequence of generations linked by descent. It also says whether or not two individuals are in the same historic sequence of generations. (Descent is thus a partial order on the set of all "people" which also partitions this set into equivalence classes called "generations").

The clearest alternative to a discrete generation model is the "continuous" generation model, in which there is no simple distinction between generations: even if by chance two particular individuals are born simultaneously, they may or may not, indeed probably will not, die simultaneously, and therefore the specific set of individuals composing the population is constantly changing.

While in discrete models, time may be considered a construct of the notion of descent, in continuous models we can only know questions of simultaneity of existence by reference to some concept of time which is independent of the descent relation.

Thus while in discrete models time can enter simply as a superscripted or subscripted index of order, in continuous models it enters as a variable, a "real time" whose existence is imagined as a continuum independent of the particular organisms under study, or as part of the physical world in which the organisms exist. Having made this presumption, we may then look at simultaneity of existence by picking particular wide or narrow bands of time by this independent measure.

On first sight, this difference between discrete and continuous models may not seem important. In demography, ecology, or population genetics the more important problems may often be modeled by either technique, and the tendency has been for development of proofs showing how good an approximation one technique is of the other (e.g., May, 1974) or showing equivalences between them, etc. However, in social theory, particularly in kinship studies and ethnological works the absence of a presumed independent "real time" has important consequences. Models which do not assume an underlying "real time" are generally more concerned with "forms", and isomorphism of forms of relations among simultaneously existing or descent sequenced sets of individuals. Thus, a cultural rule might continue to exist over a sequence of generations in which the specific individuals change. Time may be important to the existence of the individuals, but only the relations among the individuals, at all times, is important to the existence of the rule.

Concepts

I will begin by defining a generation as a set of co-existing individuals. I shall take "individuals" as a primitive term, and also for the present will take "descent" as a primitive term. Generations are connected by "descent maps", whose nature will be discussed later.

Each generation has on it a particular "structure". Depending on the concrete interpretation or on theoretical preferences, this structure may be expressed as a partition or partitions of the generation, as relations between the individuals, as a graph (or graphs) with individuals as nodes (or edges), as operators on the set of individuals or its various direct products with itself as a basis spaces, and so on. Since these various constructs usually may be translated into each other with relative simplicity, I shall here speak only of "the structure" or "the graph" of a generation in general, but in a specific case must of course specify a particular structure.

In social theory, questions of form are generally questions of selected descriptions of relations between individuals. The common language idea of "same form" has the mathematical analogues of isomorphism and homomorphism. Assuming the structures of two generations have been described in the same mathematical vocabulary, I will say the generations are isomorphic if they are of the same size (same number of individuals) and their structures are isomorphic in the sense appropriate to the selected structural description and empirical reference; I will say they are homomorphic if their sizes are different and an appropriately defined homomorphism exists between their structures. Selecting subsets of generations, and describing only the structure on the elements of the subset, then these subsets (of the same or different generations) may be isomorphic if they are the same size and their structures isomorphic, and homomorphic if they are of different sizes but also of "similar structure". Also, one may find isomorphism between equivalence classes on generations which are different sizes.

All generations which are isomorphic to one another form an isomorphism class. This class is called the state of the generation, and may be named and/or described either by a lexical reference or by displaying a "picture" of the state in graphical form, etc. While subsets of generations may form isomorphism classes, I do not call this class the state of the subset: the term "state" is reserved for description of entire generations.

One must now decide whether or not and in what way states may be connected to each other. Notice that for generations, under most concepts of descent the answer is simple: two concrete generations are either ancestral (one to the other) and are therefore connected directly by a single descent map, or connected by n maps through n - 1 intervening generations; or else the two generations are unrelated. In the first case, they are in the same sequence. When concrete examples of sequences of generations in given states occur, then we may infer that some sensible map exists between the states. However, if no concrete sequence with such a map has been presented, then we have no knowledge of the possibility of connecting states, or must use some other notion of state connectivity.

In other words, descent relations between states may only be constructed a-posteriori from knowledge of specific possible sequences of generations. This requires therefore a knowledge of what are the "rules of descent" which link generations, and how the structure of a given generation, together with the rules, constrain the possible structures of the following generation.

Now notice this connection between "time-independent" and "time-dependent" models. Time-independent models, or apparently a-temporal descriptions of social systems, are nearly universally descriptions of form, which is to say, identification of the equivalence class of states to which a particular empirical structure belongs. If one has a calculus of relations among states (which to some degree ignores the underlying specific sets of sequences implied by claiming a particular state or set of states for description of a given form) then (to that degree) one has time-independent theory. On the other hand, when speaking of particular generations and sequences, time-dependence is intimately tied to the descriptions themselves. The more precise one wishes to be, the more it is necessary to abandon the "state" descriptions and refer to specific flows in time, that is, specific sequences of generations.

Two versions of the prediction problem may now be noticed: (1) given a specific generation, predict which states may follow it; (2) given a specific state, computed how to enter it. Notice that saying how to "stay in" a particular state is computing (1) and (2) simultaneously.

Consider also these more general problems: (A) what kinds of assumptions on descent maps between generations give what possible relation/maps between states? (B) what do specific impositions of relations between states imply for relations between generations; and for the description of states themselves, principally, their size = {0, finite, infinite}; (C) what do assumptions on nature of sequences imply for states (their sizes, etc.) and for relations between states?

Thus, an adequate theory of descent ordered structures must have a broad range of capacities, but must especially include the ability to simultaneously study the "form of relationship" and the numbers or sizes specifically associated with these forms. The present chapter deals with the general problem of description and comparison of forms in various domains of culture. Appendix I develops a theory that embodies the philosophy outlined above.

The treatment suggested here differs substantially from "classical" treatments of population models, such as found in established theories in genetics or demography. In fact, structural form as such is largely ignored by those models. Few geneticists, such as Wright (1965) have, begun to exploit the necessary structural theory, while papers such as Lyubich (1971), Bertrand (1966), Gillois (1965), or Kempermann (1967), strongly suggest that an operator theory (with similar organization) may be possible and useful.

Each model below has the following properties: (1) it operates on sets, called generations -- I will in general designate these sets by G, with or without superscripts or subscripts as needed: (2) each generation has a structure S(G), which may be variously represented as a graph, a partition, etc.; (3) most, or all, of these theories are concerned with measures on the set and/or its structure -- in particular with the size, $|G| = g$, of the set and also with the sizes of various partitions of G generated by the structure; (4) most theories study sequences of generations and of measures on the generations. (The present notation and structures are also generally those of Chapter 3).

Most population theorists have been primarily concerned with point (4) or perhaps (3) of the last paragraph, and therefore have treated points (1) and (2) as only of incidental interest. This is a fundamental error, since even if the only concern is with point (4), the properties of the possible sequences are completely determined by descriptions of G and of S(G). There has unfortunately been a "natural" division of labor between those who are primarily concerned with "proper initial description" namely the choice of G and of S(G) (classically the realm of naturalists, biologists and of ethnographers) and those who are concerned with properties of technical axiom systems, that is, with the consequences of a particular choice of G and S(G) (which is the classical concern of mathematicians).

Example using category theory:

The relevant theoretical background of this section is that of Ballonoff (1973, 1974; especially 1974, Appendix II). Stated briefly, the problem is to characterize a) the possible chains and cycles of genealogical ties that may arise in a given discrete generation and b) the ways in which such configurations occurring in distinct generations may be compared and interconnected. (Much of this section appeared as Ballonoff, 1976c).

Following previous notation, let the history of a population be regarded as a succession of discrete, non-overlapping generational segments, so that the population at a particular time may be referred to as the generation G at that time. Let the network of intragenerational consanguineal and affinal ties existing at that time be called the configuration C of the given G. Although two or more of such generations are composed of wholely distinct collections of individuals, it is clear that their respective networks or configurations may nevertheless be isomorphic.

Now lump together all generations with isomorphic configurations and agree to refer to the particular configuration that characterizes any such set of generations as the state of the set. In Ballonoff (1973) reprinted here as Appendix I, and also in Ballonoff, 1976b, studied a special subset of the set of all possible states, called the set of configurational elements. This consists of certain uniquely simple network patterns, identified by a symbolic label for convenient reference. For example, M2 is a cycle formed by a pair of sibs (i.e., siblings) each married to individuals who are also sibs -- that is, a closed chain composed of 2 sibling bonds or sibships and 2 marital or mating bonds in an alternating sequence; M3 is a similar cycle with 3 marriages and 3 sibships; and M4 is the same pattern again with 4 bonds of each kind. On the other hand, the label 2(M2) represents a configuration that is composed of two independent, disconnected patterns which are both of the M2 type. Note that the number of individuals involved (the "population size") in a generation with a 2(M2) configuration is the same as in an M4, and twice that of an M2.

A concrete population history is represented in this model as a succession of discrete generations, in which adjacent generations are connected by descent. The latter notion is defined as a one-to-one map (function) that associates sibships in each generation with marriages in the preceding generation in a way that connects offspring with parents. It is at this point that a problem arises suggesting the relevance of category theory. In any given sequence of generations of a populational history the successive <u>configurations</u> are connected by a descent map, but the objects called <u>states</u> above are not necessarily connected by anything. In order to discuss the problem of relationships among states, which are configurations characterizing whole sets of generations, I shall introduce the notion of category and then attempt to construct a category of states.

DEFINITION: A category consists of

i) a set of objects, A, B, C, ...;

ii) for each pair of objects (A, B) a set [A,B], whose elements are called <u>morphisms</u> from A to B, or with <u>domain</u> A and <u>range</u> B (written $\alpha: A \to B$ or $A \overset{\alpha}{\to} B$ for $\alpha \in [A, B]$) these sets being pair wise disjoint: $(A, B) \neq (A', B')$ implies $|[A,B] \quad [A',B]| - 0$;

iii) for each triple (A, B, C) of objects a map $[A,B] \times [B,C] \to [A,C]$ or $(\alpha,\beta) \to \beta\alpha$ called <u>composition</u> of morphisms;

iv) for each object A an element $1_A \in [A,A]$ called identity morphism;

these i) to iv) subject to the restrictions

1) If $\alpha \in [A,B]$, $\beta \in [B,C]$, $\alpha \in [C,D]$, than $\gamma(\beta\alpha) = (\gamma\beta)\alpha$

2) If $\alpha \in [A,B]$ then $\alpha 1_A = \alpha$, $1_B\alpha = \alpha$.

(This definition is slightly modified from Tondeur, 1969, p. 170).

To construct a category pertinent here, let $\underline{T}_j$ be a state and consider the set $\tau = \cup\underline{T}_j$ to be the set of objects we will study. Let t_i be a particular configuration which therefore is a member of some particular state of τ. We shall say there exists a morphism $\alpha: \tau \to \tau$ from a state $\underline{T}_1 \in \tau$ to a state $\underline{T}_2 \in \tau$ if there exists a configuration $t_j \in \underline{T}_1$ and $t_i \in \underline{T}_2$ and there exists a descent map d from t_j to t_i.

Note that $\underline{T}_1 = \underline{T}_2$ is permitted, and in particular if $\underline{T}_1 = \underline{T}_2$, then we are dealing with the identity morphism $1_{\underline{T}_1}$ on $\underline{T}_1$ (and also on $\underline{T}_2$):

$$1_T : \underline{T} \to \underline{T} \ , \ \underline{T} \in \tau$$

means there exists a descent map from t_j to t_i, and t_j, $t_i \in \underline{T}$. A state with identity is called 1-stable, as is a sequence of isomorphic generations.

We can therefore identify a set $[\underline{T}_1,\underline{T}_2]$ where $\underline{T}_1$ and $\underline{T}_2$ are arbitrary states, and $[\underline{T}_1, \underline{T}_2]$ is the set of all morphisms from $\underline{T}_1$ to $\underline{T}_2$, and we can define an identity $1_T \in [T,T]$. We must now study composition of morphisms. Let $\underline{T}_1,\underline{T}_2,\underline{T}_3,\underline{T}_4 \in \tau$, and let $[\underline{T}_1, \underline{T}_2]$, $[\underline{T}_2, \underline{T}_3]$, $[\underline{T}_3, \underline{T}_4]$ be the morphisms from $\underline{T}_1$ to $\underline{T}_2$, and from $\underline{T}_2$ to $\underline{T}_3$, etc., respectively. It is certainly true that if there is $t_i \in T_1$, $t_j \in T_2$, $t_k \in T_3$, $t_n \in T_4$, then the descent maps can be noted as:

$\alpha : t_j \rightarrow t_i$
$\beta : t_j \rightarrow t_i$
$\gamma : t_j \rightarrow t_i$

and $\gamma(\beta,\alpha) = (\gamma,\beta)\alpha$ describe the same sequence. Therefore the associativity condition can be met, provided we know the descent maps exist.

Introduction of the idea of "state" seems at first innocuous, but leads to an undesirable consequence, corrected by adoption of an axiom.

THEOREM: If A is a marriage rule for which there exists a 1-stable sequence $\{\underline{G}^{(n)}\}$ satisfying A, then there exists a state $\underline{T}_A$ such that $\underline{C}^{(n)} \varepsilon \underline{T}_A$ for all $\underline{G}^{(n)} \varepsilon \{\underline{G}^{(n)}\}$.

Proof: If $\{\underline{G}^{(n)}\}$ is 1-stable, then all configurations in the sequence are isomorphic to each other, hence they belong to an isomorphism class. This class is the required state $\underline{T}_A$.

NOTE: This state clearly has an identity $1_{\underline{T}_A}$.

THEOREM: Let $\underline{T}$ be a state with an identity morphism 1. Then $|\underline{T}|$ is not finite or not ≥ 0, or there exists an individual p in some configuration $t \varepsilon \underline{T}$, and p is an ancestor of itself (and, p is a descendant of itself).

Proof: Suppose $|\underline{T}| = n$, n a finite number ≥ 0. Since descent maps are 1-1, from a particular generation $t_i \varepsilon \underline{T}$ to another generation $t_i \varepsilon \underline{T}$, after computing at most $(1_T)^{n-1}$, one reaches a generation t_1, such that t_1 is the only generation in $\underline{T}$ which has no descent map into its M-sets, so that when computing:

$$1_{\underline{T}} : t_1 \rightarrow t_k \ , \ t_1 \ , \ t_k \ \varepsilon \ T \ ,$$

k must index a generation which is a descendant of $\underline{G}_1$, otherwise we discover that $(1_T)^n$ does not exist. This violates the basic property $1_{\underline{T}} \cdot 1_{\underline{T}} = 1_{\underline{T}}$ of identify morphisms, since by this, $(1_T)^n = (1_T)^{n-1} = 1_{\underline{T}}$, which does exist.

Therefore, either there exists no such finite number n, hence no minimal sequences, or there exists an individual which is an ancestor of itself.

Now, one solution to this difficulty is to drop the notion of identity on a state, but we may preserve identities with the following axiom, and subsequent theorem, which follows immediately.

AXIOM: No individual may be an ancestor (descendant) of itself. (Every individual has ancestors.)

THEOREM: If $\underline{T}$ is a state with identity 1_T, then $|T|$ is not finite. If A is a marriage rule and $\{\underline{G}^{(n)}\}$ is a 1-stable sequence satisfying A, then $\{\underline{G}^{(n)}\}$ is an infinite sequence.

I believe that a category theory of states, sequences of states, and measures on states, all as affected by rules such as defined in Appendix I, would be extremely useful. While population models concentrate on G and on measures on G, abstract kinship theory emphasizes S(G). This is true to the point that almost the only assumption made on $|G|$ by cultural kinship models is that $|G| \neq 0$, and that not all the sub-sets induced on G by S(G) are empty. (See examples in Ballonoff, 1974b or 1974c). Studies of S(G) have been roughly of three types: relational descriptions among individuals in a given generation and in adjacent generations of a sequence; descriptions of isomorphisms of the graphs of relations within generations of a sequence. Studies of the minimal requirements on $|G|$ and associated (combinatorial) statistics required by a particular choice of S(G) and choice of descent relation have only to my knowledge been conducted by me.

Abstract Cultural Mathematics

This section explores a general framework for descriptive cultural mathematics. Concepts relied upon here were initially proposed as a method for comparing computer languages (Scott, 1970). While I retain essentially the original mathematical organization of that work, the interpretations are my own. (The term "state" in this section is not identical to that used above. I here use it more as the way kinship mathematics has used labels of kin terms to define relational trees. The implications of these two notions of state for each other would be another fruitful avenue for study.)

Consider as the basic unit of analysis a "culture" which has "within it" individuals which act both as storage units for the "elements" or "schema" of the culture of the cultural system. I here study the system of relations which exist among the cultural elements which individuals store, and which on various occasions which they disgorge, repeat, modify, forget, or act upon.

Let L be this set of "locations," and denote by ℓ an element $\ell \epsilon L$. Or, ℓ is an "individual," L the set of individuals. Let V be the set of possible units of "cultural knowledge" an individual may "contain". Then we can define a function

$$\sigma: L \rightarrow V$$

which defines for each $\ell \epsilon L$ its current "contents" $\sigma(\ell) \epsilon V$.

We can therefore consider σ as a state map of an individual since it defines equivalence classes in both L and V, and denote by S the set of all states:

$S = \{\sigma: \sigma \text{ is a state map}\}$

(It is possible to think of the state map σ as the "cultural condition" of individuals, and S as the set of "equivalent cultural conditions").

Since states can change, define a function

$\delta: S \rightarrow S$

called a state changing function, but which may leave the state unchanged as well. Notice one possibility allowed by these definitions is that there exists an assignment $\sigma(\ell)$ such that

$\sigma(\ell): S \rightarrow S$

is a state changing function, contained in the set of cultural knowledge V.

Define a procedure P as a map

$P : V \times S \rightarrow V \times S$

Now, $S = \{\sigma\} = \{L \rightarrow V\}$, and we know $\sigma(L) \varepsilon V$, so

$P : V \times V \rightarrow V \times V$

which is a map from an initial value, associated with a value stored at a particular location (e.g., given individual knowledge) to an output value and some value stored at the same initial location (individual). Notating P as a map on V X S emphasizes that we may have a whole set of possibilities in V X V denoted by one state in S.

In analysis of literary or oral history materials, consider the set V as the "corpus" to be studied, and the state function σ as the list of which individual(s) provided which variants or examples. This perspective leads to study of population distributions of objects. The same model can also describe the internal sequencing structure of particular texts, or more accurately in a particular observed body of text. In studies of kinship, consider the set V as the possible kin labels and the function σ as their assignment in a particular case. In this case S becomes the set of all simultaneous labelings σ from the perspectives of all individuals studied. (Development of this should lead to incorporation of kinship mathematics using group theory such as White, 1963). In the case of textual examples, such as riddles (as a simple example) the set S is the set of all variants which could have been studied, and therefore to study P is to study the evolution or possible forms of change (modification) which could occur. If we had a complete theory of the possible variety of types in V, we could treat study of P as the study of possible compatible and mutually accessible types.

Now consider S(G) and $|G|$ in the present context. To create a model on the lines used just above for marriage theory index the sets, V, L, and S with a "time" subscript "t" which is bigger as time goes forward, and smaller and if necessary becomes "negative" relative to a particular indexed value, as one counts backward. Consider the set $L = \cup_T G_t$ as the "whole" population including historic and future members and the set G_t as the population at time t -- that is, "the generation at time t". The state set S_t at time t consists of the observed maps

$$\sigma_t : V_t \rightarrow L_t$$

at time t, and the map P_t at time t is the map

$$P_t : V_t \times P_t \rightarrow V_{t+1} \times S_{t+1}$$

This permits study of forward but not backward movement. In some cases, either or both of the conditions

$$P_t = Q_{t+1}^{-1} \text{ or } Q_{t+1} = P_t^{-1}$$

might hold, but in general we may not assume this. For example, if Q_t here corresponds roughly to the descent maps found in appendix I, the inverse of those maps are not usually unique valued functions if applied directly to the sets L_t. Leviandier (1975) has also explicitly studied the differences in P_t and Q_t for cases of family size statistics. A very simple example of the effects of foreward versus backward operations is to realize that in any given population any pair of individuals have some common relative at some past time, but do not in general have any common descendants. Notice that we can specify that $|G_t|$ is finite, while still allowing $|L| = |\cup_t G_t|$ to possibly be infinite by considering infinite sequences of generations. Note that sequences of states such as used in the category theory discussion, could be finite and even form mathematical groups, while the underlying sequences of actual historic persons do not.

We may introduce an interesting possibility by creating the concept of an individual $\ell_t \in G_t$ having associated with it a numerical coefficient $0 \leq a_t(\ell) \leq 1$ in which the a's are essentially discontinuous functions of t, which are spontaneously 1 at the moment of the individuals "birth" or entry to puberty or social marriage, zero before that, and which decay toward zero as t increases. In the discrete model, simply set $a_\ell(t) = 1$ at time t for $\ell \in L_t$, and $a_\ell(t) = 0$ if $\ell \notin L_t(t)$. This results in a model corresponding to "ordinary demography".

We may look meaningfully at concepts having to do with "derivatives" of the various maps, functions, and even sets. Thus, the derivatives of the $a_\ell(t)$ defined in the last paragraph are related to birth/death/age schedules as studied by demography. The derivatives of the sizes of the sets G_t related to change in population size. The "derivative" of the maps σ_t show simultaneously changes in the sizes of various subsets they may induce, as well as stability of the structure $S(G_t)$ to which they relate. In fact, since in most cases the maps σ_t are simply the descriptions of the structure $S(G_t)$, the minimal structures studied in chapter 2 or Appendix I are examples where both the size-derivations and the structure derivatives of σ_t are zero!

(Study of derivations of maps may require use of a larger space. For example, one may re-define σ and S in terms of L, considered as $L = U_t G_t$, and write

$$\sigma : V_t \to L$$

$$S_t : \{\sigma : a_\ell(t) \neq 0,\ \ell \in L\}$$

so that the derivatives of σ and of $|S_t|$ make more apparent sense. An even more general conception is created by defining $V = UV_t$ and using the original form $\sigma : V \to L$ by retaining S_t as defined above. This forces study of changes in V_t in an explicit way.)

Consideration of the $a_\ell(t)$ given above also allows the possibility that for some (possibly non-integer) value of t, that the computation

$$\Sigma_\ell\ a_\ell(t) = N_t;\ a_\ell(t) \neq 0$$

gives a value for N_t which is not discrete, and in fact is <u>not</u> population size in any usual sense. Rather, N_t as given here is better interpreted as "the equivalent producing population." The derivative $\frac{dNt}{dt}$, which is just the sum of the $\frac{da(t)}{dt}$, shows the change in reproductive potential. (These concepts relate to the notion of "life mass" studied in Ballonoff, 1983 and to the geneticists concept of "effective size" of a population such as used in Chapter 5).

This concludes the chapter, and this book. I hope it does not conclude this work, as the concepts laid out here appear to contain the foundation for an empirically effective, true science, of culture.